John Wermer received his Ph.D. at Harvard in 1951, and his thesis advisor was George Mackey. His mathematical interests were strongly influenced by Arne Beurling, with whom he studied at Harvard and in Sweden. He taught at Yale from 1951 to 1954, and since then he has been at Brown. He has worked on operator theory, Banach algebras and complex function theory.

Function Algebras in the Fifties and Sixties[1]

JOHN WERMER[2]

1. INTRODUCTION

This essay is a very personal survey of a chapter of mathematical history in which I participated, the study of Function Algebras in the U.S. in the period 1950–1970. For obvious reasons the survey is very incomplete, as is the bibliography. For a balanced view of the subject the interested reader can consult three excellent works: *Introduction to Function Algebras* by A. Browder, W. A. Benjamin, Inc. (1969), *Uniform Algebras* by T. W. Gamelin, Prentice Hall, Inc. (1969), and *The Theory of Uniform Algebras* by E. L. Stout, Bogden and Quigley, Inc. (1971).

Starting in the early 1950s a band of American mathematicians went to work on some questions in complex analysis which came from two sources: the theory of polynomial approximation on compact sets in the complex plane, and the theory of commutative Banach algebras. The American mathematicians included Richard Arens at UCLA, Charles Rickart at Yale, Ken Hoffman and Iz Singer at MIT, Andy Gleason at Harvard, Hal Royden at Stanford, Errett Bishop at Berkeley, Irv Glicksberg at the University of Washington, Walter Rudin at Rochester and the University of Wisconsin, and the author at Brown. They and their students began to develop a theory of Function Algebras which formed a new link between classical Function Theory and Functional Analysis. Their inspiration came largely from the Soviet Union.

[1] A good discussion of many of the topics of this article, as well as a very extensive bibliography, is given in the article by G. M. Henkin and E. M. Čirka, *Boundary Properties of Holomorphic Functions of Several Complex Variables*, Plenum Publishing Corporation (1976).

[2] I am grateful to Andy Browder and Peter Duren for helpful comments for this article.

In the 1940s I. M. Gelfand and his coworkers had built a theory of commutative Banach algebras in which they had shown that such an algebra, if it has a unit and its radical is zero, is isomorphic to an algebra $\mathfrak{A}$ of continuous complex-valued functions on a compact Hausdorff space $\mathfrak{M}$. The points of $\mathfrak{M}$ are identified with the maximal ideals of $\mathfrak{A}$. G. Šilov had shown that among all closed subsets of $\mathfrak{M}$ there exists a smallest set $\check{S}$ with the property that if m is in $\mathfrak{M}$, then for each f in $\mathfrak{A}$

$$|f(m)| \leq \max |f(x)| \text{ taken over } \check{S}.$$

$\check{S}$ is called the *Šilov boundary* of the algebra.

A simple model for this is given by the *disk algebra* $A(D)$ consisting of all functions which are analytic in the open unit disk: $|z| < 1$ and continuous in the closed disk D: $|z| \leq 1$. Here the maximal ideal space $\mathfrak{M}$ can be identified with D and the Šilov boundary with the unit circle: $|z| = 1$. The natural norm on $A(D)$ is given by $\|f\| = \max |f(z)|$, taken over D.

The question arises: let $\mathfrak{A}$ be an arbitrary semi-simple commutative Banach algebra with unit, such that $\check{S}$ is nontrivial, i.e., $\check{S}$ is strictly smaller than $\mathfrak{M}$. Does there exist an *abstract function theory* for $\mathfrak{A}$, i.e., do the functions in $\mathfrak{A}$ behave on $\mathfrak{M}\backslash\check{S}$ like analytic functions (as in the example of the disk algebra)? Furthermore, does $\mathfrak{M}\backslash\check{S}$ possess *analytic structure*, i.e., can we find subsets of $\mathfrak{M}\backslash\check{S}$ which can be made into complex manifolds on which the functions in $\mathfrak{A}$ are analytic? If enough such analytic structure could be shown to exist, this would explain the Šilov boundary in terms of the maximum principle of analytic function theory.

In 1952 a brilliant achievement by the Soviet Armenian mathematician S. N. Mergelyan provided a second source of inspiration. Mergelyan showed in **[48]** that if X is a compact set in the z-plane $\mathbf{C}$ such that $\mathbf{C}\backslash X$ is connected, then every function which is continuous on X and analytic on the interior of X can be uniformly approximated on X by polynomials in z. This result can be read as a statement about a certain Banach algebra. We let $P(X)$ denote the uniform closure on X of the polynomials in z and we put on $P(X)$ the supremum norm over X. Then $P(X)$ is a Banach algebra, the maximal ideal space $\mathfrak{M}$ coincides with X, and the Šilov boundary $\check{S}$ coincides with the topological boundary of X. Mergelyan's theorem yields that a function φ defined and continuous on $\mathfrak{M}$ belongs to $P(X)$ if and only if φ is analytic on $\mathfrak{M}\backslash\check{S} = \text{int}(X)$ in the natural analytic structure which $\text{int}(X)$ inherits from $\mathbf{C}$.

2. Uniform Algebras

For the problems mentioned above, of constructing an abstract function theory for $\mathfrak{A}$ and of exhibiting analytic structure on $\mathfrak{M}\backslash\check{S}$, it seemed natural to take the norm on the algebra $\mathfrak{A}$ to be a uniform norm. The "Function

Algebras" to be studied where then as follows: we fix a compact Hausdorff space X and an algebra $\mathfrak{A}$ of continuous functions on X such that $\mathfrak{A}$ is closed in the algebra $C(X)$ of all continuous functions on X, contains the constants, and separates the points of X. If we put on $\mathfrak{A}$ the uniform norm over X, $\mathfrak{A}$ is then a Banach algebra. $\mathfrak{M}$ is a compact space in which X lies embedded, as proper subset in general, and $\check{S}$ is a closed subset of X.

Such algebras were baptised *uniform algebras* by Errett Bishop in 1964. He thought the name sounded good, and it has stuck. One says that $\mathfrak{A}$ is a uniform algebra *on* X. Uniform algebras are plentiful in nature. Here are some examples:

(i) Let Y be a compact set in $\mathbb{C}^n$, the space of n complex variables. Let $P(Y)$ denote the uniform closure on Y of polynomials in the complex coordinates $z_1, \ldots, z_n$. Then $P(Y)$ is a uniform algebra on Y.

The disk algebra is a special case. For $n = 1$ and so $Y \subset \mathbb{C}$, Mergelyan's theorem tells us which functions belong to $P(Y)$.

(ii) Let Σ be a finite Riemann surface with boundary $\partial\Sigma$ and denote by $A(\Sigma)$ the algebra of functions continuous on Σ and analytic on $\Sigma \backslash \partial\Sigma$. $A(\Sigma)$ is a uniform algebra on Σ.

(iii) Let K be a compact set in $\mathbb{C}$ and let $R_0(K)$ denote the space of rational functions whose poles lie in $\mathbb{C} \backslash K$. Let $R(K)$ denote the uniform closure of $R_0(K)$ on K. Then $R(K)$ is a uniform algebra on K.

(iv) Let H^∞ denote the algebra of all bounded analytic functions on the open unit disk. By Fatou's theorem, H^∞ is embedded in L^∞ of the unit circle, and L^∞, in turn, is isomorphic to $C(X)$ for a (complicated) space X. H^∞ is a uniform algebra on X.

(v) The Stone–Weierstrass theorem yields that the only uniform algebra on a compact space X which is closed under complex conjugation is the full algebra $C(X)$.

A first indication that it might be possible to do abstract function theory on a uniform algebra A was the proof that *representing measures* always exist. By a representing measure for a point m in $\mathfrak{M}$ is meant a probability measure μ on the Šilov boundary $\check{S}$ such that for all f in A

$$f(m) = \int f \, d\mu.$$

Arens and Singer in [5] and John Holladay in his Yale thesis (1953) proved that such a μ exists.

In the case of the disk algebra $A(D)$, μ is unique for a given m and is the Poisson measure on the circle, corresponding to m. In general, μ is far from unique.

A representing measure μ is multiplicative on A, i.e.,

$$\int fg\,d\mu = \left(\int f\,d\mu\right)\cdot\left(\int g\,d\mu\right) \quad \text{for all } f, g \text{ in } A,$$

and conversely, each multiplicative probability measure is the representing measure for some point m in $\mathfrak{M}$.

In 1953 in [64] Šilov made another fundamental contribution to Banach algebra theory by introducing the use of analytic functions of several complex variables into the theory. Let $\mathfrak{A}, \mathfrak{M}$ be as above. Suppose that $\mathfrak{M}$ is disconnected, i.e., $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$ where $\mathfrak{M}_1, \mathfrak{M}_2$ are disjoint closed sets. Šilov showed that $\exists\, e$ in $\mathfrak{A}$ with $e^2 = e$ such that $e = 1$ on $\mathfrak{M}_1$ and $e = 0$ on $\mathfrak{M}_2$.

Not long after, Arens and Calderon in [4] and L. Waelbroeck in [68] developed a functional calculus for analytic functions of n variables acting on n-tuples of elements of a commutative Banach algebra.

Another application of several complex variables to Banach algebra theory was the algebraic description of the first cohomology group of the maximal ideal space, independently by R. Arens in [3] and H. Royden in [58]. They showed that for $\mathfrak{A}, \mathfrak{M}$ as above, $H^1(\mathfrak{M}, \mathrm{Z})$ is isomorphic to the quotient group of the group of units of $\mathfrak{A}$ by the subgroup of elements $\exp(y)$ with y in $\mathfrak{A}$.

3. Gleason's Program

Andrew Gleason launched the earliest attacks on the problem of analytic structure in the maximal ideal space of a uniform algebra.

In the case of the disk algebra $A(D)$ those maximal ideals m corresponding to an interior point of the disk, say the point a, have the algebraic property of being *simply generated*: every f in the ideal m can be written in the form: $f = g(z - a)$ with g in $A(D)$. Maximal ideals corresponding to boundary points of D are not simply generated. Gleason obtained the following striking result: *Let A be a uniform algebra and fix m in $\mathfrak{M}$. Suppose that the ideal m is finitely generated in the algebraic sense. Then some neighborhood U of m in $\mathfrak{M}$ can be given the structure of an analytic variety such that every h in A is analytic on U.*

He lectured on this result in the mid-fifties, and published it in [29].

In another direction, Gleason observed the following: with $A, \mathfrak{M}$ as before, let m_1, m_2 be two points in $\mathfrak{M}$. Then $|f(m_1) - f(m_2)| \le 2$ whenever f belongs to the unit ball of A. It may happen that there exists $k < 2$ such that $|f(m_1) - f(m_2)| \le k$ whenever f belongs to this unit ball. In the case of the disk algebra, this occurs whenever m_1 and m_2 lie in the open unit disk. This suggests the following general definition: for m_1, m_2 in $\mathfrak{M}$, put $m_1 \sim m_2$ whenever $\exists$ such a $k < 2$, or, in other words, whenever the distance from m_1 to m_2 in the dual Banach space of A is less than 2. Gleason showed that $\sim$ is an equivalence relation on $\mathfrak{M}$. (Since $2+2=4$, the transitivity of the relation $\sim$ is not evident!) He called the equivalence classes under $\sim$ *the parts of* $\mathfrak{M}$.

For the case of the disk algebra $A(D)$, the open unit disk is one part and each point on the unit circle is a one-point part. For the case of the bi-disk algebra $A(D^2)$ which consists of all functions which are continuous on the closed bi-disk $D^2 = \{|z| \leq 1\} \times \{|w| \leq 1\}$ in $\mathbb{C}^2$ and analytic on the open bi-disk, the maximal ideal space $\mathfrak{M} = D^2$, and the parts are as follows: the open bi-disk is one part, each disk: $z = z_0, |w| < 1$ and each disk: $|z| < 1, w = w_0$ with $|z_0| = 1$ and $|w_0| = 1$ is a part; the remaining parts are the one-point parts on the distinguished boundary $\{|z| = 1\} \times \{|w| = 1\}$ of D^2. Thus the parts here are complex manifolds of dimensions 2, 1, and 0.

Gleason lectured on these ideas, [28], at the Conference on Analytic Functions at the Institute for Advanced Study in Princeton in September, 1957. This was a marvelous meeting. The people there interested in Banach algebras included R. Arens, R. C. Buck, L. Carleson, A. Gleason, K. Hoffman, S. Kakutani, Lee Rubel, H. Royden, I. Kaplansky, L. Waelbroeck, and myself. Many of the giants of function theory gave talks, both on one and several complex variables, and tolerated those of us who didn't know much about either one or several complex variables. The two weeks of the conference were for us enormously stimulating and provided the germ of much later work on Function Algebras.

Kakutani had studied H^∞ as a Banach algebra, and reported on his work in [41]. At the conference, he discussed the boundary behavior of a bounded analytic function in terms of normed ring theory, [42].

Earlier, Kakutani had raised the following basic question about H^∞ as a ring: the open unit disk is naturally embedded as an open subset Δ of the maximal ideal space $\mathfrak{M}$ of H^∞, and so its closure $\overline{\Delta}$ is contained in $\mathfrak{M}$. The set $\mathfrak{M}\backslash\overline{\Delta}$ was called the "Corona".

Is the Corona empty, i.e., is Δ *dense in* $\mathfrak{M}$? Suppose that the answer is "Yes" and consider an n-tuple of functions f_j in H^∞ with $\sum_{j=1}^n |f_j| \geq \delta$ on Δ, where δ is a positive constant. Then $\sum_{j=1}^n |f_j| \geq \delta$ on $\mathfrak{M}$ and so the f_j have no common zero on $\mathfrak{M}$. Hence the ideal generated by the f_j is contained in no maximal ideal of H^∞ and so is the whole ring. It follows that there exist g_j in H^∞, $j = 1, \ldots, n$, such that

$$\sum_{j=1}^{n} f_j g_j = 1.$$

The problem of the existence of the g_j under the given assumption on the f_j turned out to be a very deep problem. This "Corona problem" was solved by Lennart Carleson in [22], and it follows that the Corona is indeed empty. Carleson's result and his method of proof has had a major impact on analysis. All this is treated in John Garnett's book mentioned in Section 7 below.

A breakthrough in the understanding of the maximal ideal space of H^∞ occurred at the conference, in the form of the birth of I. J. Schark, [62]. Schark's paper exhibited analytic structure in $\mathscr{M}\backslash\Delta$ for the first time. Schark never published again, since his name was put together from the initials of participants at the conference. So Schark did not perish; he vanished.

In his talk, Gleason formulated the following *Conjecture*: *Let m_1, m_2 be two points in the maximal ideal space $\mathfrak{M}$ of a uniform algebra. Then a necessary and sufficient condition for m_1 and m_2 to be in the same part of $\mathfrak{M}$ is that m_1 and m_2 can be connected by a finite chain of analytic images of the unit disk, contained in $\mathfrak{M}$.* A second idea Gleason introduced in [28] was the notion of a *Dirichlet Algebra.* The real parts of the functions belonging to a uniform algebra A on a space X can be viewed as "harmonic" on $\mathfrak{M}\backslash X$, as can uniform limits on $\mathfrak{M}$ of sequences of such functions. Gleason called A a *Dirichlet Algebra on X* if every real continuous function on X is the restriction to X of such a harmonic function, or, equivalently, if the real parts of functions in A form a uniformly dense subspace of the real continuous functions on X.

The disk algebra $A(D)$ may be viewed as a uniform algebra on the circle $|z| = 1$, with norm the supremum norm on the circle, rather than as a uniform algebra on the disk. $A(D)$ is a Dirichlet algebra on the circle.

Gleason wrote in [28] about Dirichlet algebras: "It appears that this class of algebras is of considerable importance and is amenable to analysis." It turned out subsequently that this preliminary judgment was right on target. At the time, in September 1957, Gleason's ideas were sufficiently strange and novel that I (and many of us, I imagine) did not fully grasp their significance.

4. The Summer of 1959 in Berkeley

In the summer of 1959 a lot of people working on Functional Analysis gathered, rather informally, in Berkeley. My wife Kerstin and I took our two boys, two and five years old, put them in our Chevy and drove across the country. It had been hot when we left the East Coast and got steadily hotter as we drove west until suddenly, as we came into Berkeley, a discontinuity occurred and we were in a cool and lush paradise, the sky blue, the air balmy, and all garden flowers blooming wildly.

I had along with me a recent paper by Henry Helson and David Lowdenslager, [33], in which they studied certain spaces of functions given by Fourier series on the torus. Earlier, Arens and Singer in [6], and Mackey in [47], had given a group-theoretic approach to analytic functions, based on the following observation: A Fourier series $f(x) = \sum_n c_n e^{inx}$ on the unit circle is the boundary function of a function analytic in the unit disk if and only if $c_n = 0$ for $n < 0$. Replacing the circle by the torus, one may consider Fourier series $f(\vartheta, \varphi) = \sum_{n,m} c_{nm} e^{in\theta} e^{im\varphi}$ in two variables. One specifies a half-plane S in the lattice $\mathbf{Z}^2$ and regards f to be "analytic", relative to S, if $c_{nm} = 0$ outside of S. An interesting example is obtained by taking S to be the set of points (n, m) in $\mathbf{Z}^2$ with $n + m\alpha \geq 0$, where α is a fixed irrational number. Helson and Lowdenslager showed in [33] that a series of classical boundary value theorems of function theory have counterparts for functions "analytic relative to S". Their results were dramatic and their proofs made elegant use of L^2-methods. Their paper stirred Solomon Bochner's interest, as he had looked at related questions at an earlier time. He showed that their proofs depended only on two properties: first, that for fixed S the class of S-analytic functions continuous on the torus is an algebra, and second, that the real parts of the functions in this algebra are dense in the real continuous functions on the torus. The group structure on the torus entered only through these properties. So Bochner, quite independently of Gleason, was led to the same Dirichlet algebras [19]. Thus it turned out that certain basic results about boundary-functions of analytic functions in the disk remain true, when properly stated, for an arbitrary Dirichlet algebra. How does this look?

For the case of the disk algebra, the measure $\frac{1}{2\pi}\,d\theta$ is the representing measure for the origin. For $p \geq 1$,the Hardy space H^p is defined as the closure of $A(D)$ in L^p on the circle with respect to this measure. Let now A, on X, be a Dirichlet algebra and fix m in $\mathfrak{M}$. Let μ be the unique representing measure for m on X, for the algebra A. We define $H^p(\mu)$ as the closure of A in $L^p(X, \mu)$. For f in $H^p(\mu)$, $f(m)$ is defined as $\int f\,d\mu$. One then has, for instance, the following:

THEOREM 1. *Let A, m, μ be as above. Fix a nonnegative function w on X which is summable with respect to μ. A necessary and sufficient condition for w to have a representation*

$$w(x) = |f(x)|^2 \text{ a.e.-}d\mu \text{ on } X$$

for some f in $H^2(\mu)$ with $f(m) \neq 0$ is that

$$\int \log w \cdot d\mu > -\infty.$$

THEOREM 2. *Let W be a closed subspace of $H^2(\mu)$ invariant under multiplication by elements of A, i.e., such that $f\varphi \in W$ whenever $\varphi \in W$ and $f \in A$. Assume also that 1 is not orthogonal to W. Then there exists a bounded function E_0 in W with $|E_0(x)| = 1$ a.e.-$d\mu$ such that*

$$W = \{E_0 g \mid g \in H^2(\mu)\}.$$

Theorems 1 and 2, in the case when A is the disk algebra, are classical results of, respectively, Szegö and Beurling.

When I realized, in Berkeley, how all these things fitted together I got quite excited. John Kelley and Errett Bishop had been studying Dirichlet algebras, and tutored me in the subject, and I also had the benefit of talking to Helson about his work with Lowdenslager. So I was able to prove the truth of Gleason's conjecture about parts, for the case of Dirichlet algebras, in the following form: *Let A be a Dirichlet algebra,* $\mathfrak{M}$ *its maximal ideal space and P a part of* $\mathfrak{M}$. *Then either P is a single point, or P is an analytic disk, i.e., P is the one-one image of the disk* $|\lambda| < 1$ *by a continuous map* ψ *such that* $h \circ \psi$ *is analytic on* $|\lambda| < 1$ *for each h in A* [72].

When we left Berkeley to go home at the end of August, we ran into several people at gas stations and so on, whom we had met upon arriving, who had noted the Rhode Island plates on our car and had told us that they themselves came from the East. When they realized we were going back, they were amazed: "You've seen California and you're going back East!" they said. My five-year-old son said, "Let's go home to America!" (meaning Providence, Rhode Island).

5. Errett Bishop and the General Theory of Uniform Algebras

Dirichlet algebras were almost too good to be true. The general uniform algebra is much less tractable, largely due to the nonuniqueness of representing measures for fixed points m in $\mathfrak{M}$. However, a series of results about general uniform algebras was discovered, with important applications to many questions in analysis. In this general theory, the unquestioned leader was Errett Bishop. Bishop was on the faculty at Berkeley from 1954 to 1965 and then on the faculty of the University of California at San Diego until his untimely death in 1983.

He was one of the most remarkable people I have known. He was a mathematician of amazing insight and penetration, absolutely fearless and with a profound commitment to mathematics. In his last years he was somewhat isolated in the mathematical community, because of his absolute dedication to constructive methods in mathematics.

In the period about which I am writing, Bishop's work and personal contact with him was enormously stimulating to the rest of us, and led to much work by other people, both jointly with him and independently of him. There was the famous joint work by Bishop and Karel de Leeuw on the Choquet boundary and by Bishop and Phelps on Banach spaces. Stolzenberg and Bishop worked closely together on polynomially convex hulls, as did Rossi and Bishop on problems about complex manifolds. My own work on analytic structure in maximal ideal spaces, e.g. in the joint paper [7] with Aupetit, and work on the same problem by Gamelin in [27], grew out of Bishop's rich paper [15]. And so on.

Here I can only mention a few of Bishop's contributions to the general theory of function algebras. The interested reader is referred to [16], [18],

[56] for more extensive discussions of his work. Further, his collected papers appear in [76].

(i) *Peak points.* A point x_0 in X is called a *peak point* for the uniform algebra A on the space X if $\exists f$ in A with $f(x_0) = 1$ and $|f(x)| < 1$ on $X \setminus \{x_0\}$. In **[11]** Bishop showed that for X metrizable peak points exist and the set M of all peak points is a *minimal boundary* for A in the sense that for each g in A there is some point x in M with $g(x) = \|g\|$, and, of course, no proper subset of M has this property. It follows that M is a dense subset of the Šilov boundary $\check{S}$. Moreover, if m is a point of $\mathfrak{M}$ there exists a representing measure for m which lies on M.

The existence of peak points had earlier been observed by Gleason (unpublished).

Bishop was able to apply the notion of peak point to the problem of rational approximation. Let X be a compact subset of $\mathbb{C}$. As in Section 2 above, we write $R(X)$ for the uniform closure on X of those rational functions which are analytic on X. When does $R(X) = C(X)$? Clearly, for this to happen the interior of X must be empty. When X has connected complement in $\mathbb{C}$, Mergelyan's theorem shows that this is also sufficient. However, in general the condition is not sufficient, and to show this Mergelyan in 1952 in **[48]** constructed the following set S: remove from the closed unit disk $|z| \leq 1$ a countable family of disjoint open disks: $|z - a_j| < r_j$, $j = 1, 2, \ldots$ such that $\sum_j r_j < \infty$, and denote by S the closed set that remains. By Cauchy's theorem, the complex measure dz on the union of the circles $|z - a_j| = r_j$ together with the unit circle annihilates $R(S)$, and hence $R(S) \neq C(S)$. If A_j, r_j are chosen so that the interior of S is empty, we have the desired example. For obvious reasons, S is called a *Swiss Cheese*. It turned out that, in fact, Mergelyan had *rediscovered* the Swiss Cheese; in 1938 the Swiss mathematician Alice Roth had given such an example. The Swiss Cheese has been very useful to people constructing counterexamples in the study of Function Algebras. My colleague Bob Accola told me that Function Algebras *is* the study of the Swiss Cheese, but this is not strictly correct.

Let now X be an arbitrary compact subset of $\mathbb{C}$. Bishop showed the following: $R(X) = C(X)$ *if and only if each point x in X is a peak point for the algebra* $R(X)$. An extension of this result was found by Donald Wilken in **[74]**. A peak point is always a one-point part of the maximal ideal space. For $R(X)$ the maximal ideal space is precisely X. Wilken showed that *each part of X is either a one-point part, or has positive 2-dimensional Lebesgue measure.*

(ii) *The antisymmetric decomposition.* If A is a uniform algebra on X, a subset Y of X is called a *set of antisymmetry* if every function in A which is real-valued on Y is constant on Y. As example we may take X to be the solid cylinder $\{|z| \leq 1\} \times \{0 \leq t \leq 1\}$ and A to be the algebra of all continuous functions on X which are analytic on each slice: $t = t_0$, $|z| < 1$. Then each disk: $t = t_0$, $|z| \leq 1$ is a set of antisymmetry. For a general uniform algebra A on X Bishop showed in **[13]**: *Let $\{Y_\alpha\}$ be the family of all maximal sets of antisymmetry. Then the Y_α give a closed partition of X and a continuous function f on X belongs to A if and only if each restriction $f|_{Y_\alpha}$ belongs to the restriction $A|_{Y_\alpha}$.*

If the Y_α are the points of X, one recovers the Stone–Weierstrass theorem. A less complete result had been obtained earlier by Šilov, **[63]**. Bishop's result reduces the study of general uniform algebras to the study of *antisymmetric* such algebras, i.e., uniform algebras which contain no nonconstant real-valued function.

(iii) *Jensen measures.* The representing measure $\frac{d\theta}{2\pi}$ for the origin for the disk algebra $A(D)$ satisfies Jensen's inequality:

$$\log|f(0)| \leq \int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi}$$

for each f in $A(D)$.

Let A be a uniform algebra and fix m in $\mathfrak{M}$. Can a representing measure μ be found for m which satisfies such an inequality? Arens and Singer had shown this to be true in certain cases. In **[15]** Bishop showed it in general: *Let A be a uniform algebra, m a point of $\mathfrak{M}$. There exists a representing measure μ for m such that*

$$\log|f(m)| \leq \int \log|f|\, d\mu$$

for each f in A.

Such a measure μ is called a *Jensen measure* for m. Jensen measures have turned out to be very useful.

6. Irving Glicksberg and Orthogonal Measures

Let A be a uniform algebra on the space X. A complex measure ν on X is called *orthogonal* to A if

$$\int f\, d\nu = 0 \text{ for every } f \text{ in } A.$$

We write $A^\perp$ for the family of all such measures. If we know $A^\perp$, then we can tell, using the Hahn–Banach theorem, whether a given function h in $C(X)$ belongs to A: $h \in A$ if and only if

$$\int h\, d\nu = 0 \text{ for each } \nu \text{ in } A^\perp.$$

The classical theorem of F. and M. Riesz identified all the orthogonal measures for the disk algebra. Frank Forelli's work in **[25]** gave a function-algebraic approach to this result.

In a series of papers [30], [31], [32], the last jointly with me, Glicksberg analyzed measures orthogonal to a uniform algebra. He applied his results to obtain elegant new proofs of Bishop's results on general uniform algebras, as well as to problems in interpolation, approximation, and so forth.

In [10] and [12] Bishop considered a compact set X in $\mathbb{C}$ with $\mathbb{C}\backslash X$ connected and looked at the measures ν on X orthogonal to the algebra $P(X)$. He showed that such a measure ν always arises from a certain analytic differential $g(z)dz$ on the interior of X. In [32] Glicksberg and I adapted these ideas to Dirichlet algebras. Let A be a Dirichlet algebra on a space X. For each m in $\mathfrak{M}$, let λ be the representing measure for m and let $H^1(\lambda)$, as earlier, denote the closure of A in $L^1(X,\lambda)$. If $k \in H^1(\lambda)$ and $\int k \cdot d\lambda = 0$, then $k \cdot d\lambda$ is orthogonal to A, since if f is in A,

$$\int f(k d\lambda) = \left[\int f \cdot d\lambda\right]\left[\int k \cdot d\lambda\right] = 0.$$

Hence we get "obvious" orthogonal measures for A by forming convergent series

$$\sum_i k_i \cdot d\lambda_i$$

where each λ_i is a representing measure and $k_i \in H^1(\lambda_i)$ and $\int k_i d\lambda_i = 0$. We showed that every complex measure ν in $A^\perp$ has a representation

$$\nu = \sum_i k_i \cdot d\lambda_i + \sigma, \tag{1}$$

with k_i, λ_i as above and such that σ is orthogonal to A and is singular with respect to every representing measure for A.

As an application, we took a compact plane set X with connected complement and boundary ∂X, and took $A = P(X)$. By the classical Walsh–Lebesgue theorem, $P(X)$ is a Dirichlet algebra on ∂X. In this case, one can show that every measure σ appearing in (1) vanishes. (1) then quickly implies Mergelyan's theorem on polynomial approximation on X mentioned earlier.

Lennart Carleson in [23] gave an ingenious new proof of the Walsh–Lebesgue theorem, and went on to give a proof of Mergelyan's theorem, also based on Bishop's ideas.

Irv Glicksberg was an unusual person. He was ever cheerful, with unlimited enthusiasm and unfailing generosity. He enjoyed every bit of good mathematics that he met up with, and it usually stimulated new ideas in him. He was a delightful, indefatigable correspondent, a fanatic photographer, and fond of jaunty headgear. Politically, he was a staunch liberal, and so he found plenty to get mad about in the last twenty years. On most other questions he had a tolerant point of view.

I was planning to spend a year at the University of Washington with him in 1983, when I was shocked to hear of his death.

7. Function Algebras at MIT and at Brown

In the late fifties and early sixties, Iz Singer and Ken Hoffman presided over a very fruitful mathematical activity at MIT. Their students during that time included Andrew Browder, Hugo Rossi, and Gabriel Stolzenberg, each of whom made important contributions to the study of Function Algebras.

Hoffman and Singer jointly in [38] answered a series of questions on Function Algebras which had been posed by Gelfand. In [37] and [39] they studied *maximal* uniform algebras on a space X, i.e., algebras A such that if B is a closed subalgebra of $C(X)$ which contains A, then either $B = A$ or $B = C(X)$.

A major open problem at that time was to prove a *local maximum modulus principle* for function algebras. If z_0 is a point in the domain of analyticity of a function F and U is a neighborhood of z_0, then $|F(z_0)| \leq \max |F|$ taken over the boundary of U. The corresponding statement for a uniform algebra A with maximal ideal space $\mathfrak{M}$ should be this: fix m in $\mathfrak{M}$ and let U be a neighborhood of m whose closure lies in $\mathfrak{M}\backslash\check{S}$. Then $|f(m)| \leq \max |f|$ taken over the boundary of U, whenever $f \in A$. *Is this true*? We all tried to prove this, but, lacking insight into several complex variables, we had no luck. At last Hugo Rossi showed how to do it. I remember the excitement of a late evening phone call, Singer to Rossi when Rossi was in Princeton, where he told us about his proof. The secret was a clever use of the solution of the Cousin problem in n complex variables. Rossi's paper on this is [55], in 1960. Much of what has been found about uniform algebras since then has depended on this local maximum modulus principle.

The local maximum modulus principle, as well as various examples of maximal ideal spaces which had been worked out in the meantime, as well as Gleason's conjecture about parts, all encouraged an effort to prove, in general, the existence of analytic structure in $\mathfrak{M}\backslash\check{S}$. One way to test this question was to look at examples in n complex variables. Let X be a compact set in $\mathbf{C}^n$ and let $P(X)$ be defined as in example (i) in Section 2 above. The maximal ideal space $\mathfrak{M}$ of the uniform algebra $P(X)$ has a natural identification with the so-called *polynomially convex hull* $\hat{X}$ of X, which had come up in the 1930s in the work of K. Oka, [51] and [52]. $\hat{X}$ consists of all points $z^0 = (z_1^0, \ldots, z_n^0)$ in $\mathbf{C}^n$ such that

$$|Q(z^0)| \leq \max |Q| \text{ over } X$$

for every polynomial Q on $\mathbf{C}^n$.

If analytic structure exists on $\mathfrak{M}\backslash\check{S}$, then there must be complex analytic varieties contained in $\hat{X}\backslash X$. Gabriel Stolzenberg in the winter of 1960–1961 constructed a set X on the boundary of the bi-disk: $|z_1| \leq 1$, $|z_2| \leq 1$ in $\mathbf{C}^2$ such that neither one of the coordinate projections $z_1(\hat{X})$ and $z_2(\hat{X})$ contains any open subset of the plane, while at the same time $\hat{X}$ contains the point $(0,0)$ and hence is larger than X. Then $\hat{X}$ contains no analytic variety, for else $\hat{X}$ would contain some proper analytic disk Δ and then either $z_1(\Delta)$ or $z_2(\Delta)$ would have nonvoid interior [65].

Thus the hope for analytic structure in $\mathfrak{M}\backslash\check{S}$ in the general case was gone forever. It was a heavy blow. From the perspective of today, almost thirty years later, I should say that Stolzenberg's example taught us that the story of polynomially convex hulls is much subtler than we had thought, but that some satisfactory understanding of these hulls is starting to emerge at the present time.

Stolzenberg himself made other incisive studies of polynomially convex hulls in the sixties, in [66] and [67].

In addition to the people just mentioned, MIT had in this period a number of junior faculty and academic visitors working on Function Algebras and related matters. These included Stephen Fisher, Ted Gamelin, John Garnett, Eva Kallin, and Donald Wilken. There was lively interaction between the Analysis Seminar at Brown, run by Andy Browder and myself, and these MIT people. Hoffman and Singer were good friends of mine, and much of my own work arose from conversations with them and others of the group.

Once, after Ken Hoffman and I had finished a particularly long-lasting and noisy mathematical conversation at my home in Providence, my three-and-a-half-year-old son came into the room, waving his arms and spouting a stream of nonsense syllables. "I am talking mathematics!" he told us.

Among the Ph.D. students working on Function Algebras who wrote their theses at Brown were Andy Browder and Robert McKissick, both borrowed from MIT; further Mike Voichick, John O'Connell, Bernie O'Neill, and Richard Basener (my students), Al Hallstrom, Jim Wang, and Kenny Preskenis (Browder's students), and Tony O'Farrell (Brian Cole's student). Stu Sidney (Gleason's student) and Lee Stout (Rudin's student) were part of this same mathematical generation, as were Mike Freeman and Laura Kodama, Bishop's students. H. S. Bear (John Kelley's student) is in this group, and Barney Weinstock (Hoffman's student) came somewhat later. Larry Zalcman was an MIT graduate student in this period. He wrote the volume *Analytic Capacity and Rational Approximation* [75], which gave a very valuable exposition in English of the recent work of Vituškin and his school on the algebra $R(X)$, example (iii) in Section 2 above.

Andy Browder joined the Brown department in 1961. One result of Browder's thesis concerned the topology of *polynomially convex sets*, i.e., sets X which coincide with their polynomially convex hull: *let X be such a compact set in* $\mathbf{C}^n$. Then the kth Čech cohomology of X with complex coefficients vanishes for $k \geq n$ [20]. It follows in particular that if Y is a compact orientable n-manifold in $\mathbf{C}^n$, then $\hat{Y}$ is larger than Y. Identifying the set of "new points" $\hat{Y}\backslash Y$ has turned out to be a difficult problem, only partially solved even for 2-manifolds in $\mathbf{C}^2$.

Eva Kallin joined the Brown faculty in 1965. B. Weinstock and G. Stolzenberg also taught at Brown for some years in the sixties. Kallin had written her thesis at Berkeley, with J. L. Kelley, and in it she had solved the following famous problem: *if a function belongs locally to a uniform algebra A, must it belong to A*? More precisely, if A is a uniform algebra and if a function f continuous on $\mathfrak{M}$ has the property that each point m in $\mathfrak{M}$ has a neighborhood U such that $f|_U = F|_U$, for some F in A, does then f belong to A? Kallin [43] gave a counterexample. G. Šilov who had earlier published an erroneous proof of the result, sent her a congratulatory postcard. Another result of Kallin's concerned the "n balls problem": consider n closed disjoint balls $B_1, \ldots, B_n$ in $\mathbb{C}^N$. Is their union polynomially convex? For $n = 1$ or 2 one sees at once that the answer is "Yes". For $n = 4$ the answer is unknown as of today. Kallin showed in [44] that the answer is "Yes" for $n = 3$.

One other major line of research at MIT at that time was Hoffman's work on the algebra H^∞ of bounded analytic functions on the unit disk. H^∞ is a uniform algebra. Its maximal ideal space, $\mathfrak{M}(H^\infty)$, is as mysterious a compact space as an analyst is likely to encounter. In his book *Banach Spaces of Analytic Functions*, which was published in 1962 by Prentice-Hall, Hoffman devoted Chapter 10 to H^∞ as a Banach algebra.

That book as a whole was a milestone. It showed to the world of classical analysts and to the world of functional analysts that they were brothers and sisters rather than strangers (as many had thought). One source of this recognition was for Hoffman, as it was for myself and many others, the towering figure of Arne Beurling who in his own work had combined classical and abstract analysis in essentially new ways.

One observation which Hoffman made was that H^∞ is *almost*, but not quite, a Dirichlet algebra on its Šilov boundary $\check{S}(H^\infty)$. For a uniform algebra A on a space X write $\log|A^{-1}|$ for the space of all functions $\log|f|$ such that f and f^{-1} both belong to A. Hoffman called A *logmodular* if $\log|A^{-1}|$ is uniformly dense in the real continuous functions on X. Dirichlet algebras are logmodular (trivially), but not conversely. Logmodular algebras still enjoy the property that representing measures for points in $\mathfrak{M}$ are unique. H^∞ is a logmodular algebra on $\check{S}(H^\infty)$. In [35] Hoffman developed the theory of logmodular algebras and showed that they enjoyed almost all the pleasant properties of Dirichlet algebras. In particular, their Gleason parts were either points or analytic disks. This last result raised the question of describing the Gleason parts of H^∞ explicitly. In the paper [36] Hoffman solved this very difficult problem, making use of the deep work of Lennart Carleson in [21] and Donald Newman in [49].

H^∞ as a Banach algebra and, in particular, as a subalgebra of L^∞ on the circle has, since Hoffman's work, been the subject of intensive investigation. This theory is closely connected with the modern theory of bounded linear operators on a Hilbert space. An exposition of the work on H^∞ from a function-theoretic point of view is found in John Garnett's book *Bounded Analytic Functions*, published by Academic Press in 1981. In particular, this book treats the Corona Problem, mentioned in Section 3 above, including the remarkable new solution of the problem by Tom Wolff.

A further development in the abstract direction came in the work of Lumer in [46], where Lumer makes as his only hypothesis on a uniform algebra the uniqueness of the representing measures for the points of $\mathfrak{M}$. Other extensions of this theory are given by P. Ahern and D. Sarason in [1] and by K. Barbey and H. König in [8].

8. YALE

I taught at Yale from 1951 to 1954 and at Brown after that. Yale provided a superb environment for a young analyst. The senior people in analysis, Rickart, Kakutani, Dunford, and Hille were very active, friendly and encouraging, and the junior people, Jack Schwartz, Henry Helson, Bill Bade, Bob Bartle, Frank Quigley, and myself had a very lively time in the analysis seminar. We all taught calculus, Math 12, out of Ed Begle's book, which is based on the axioms of the real number system. The combination of axioms, Yalies, and ourselves made a heady brew. Our wives were sick of conversations about Math 12 which went on at all department parties. The normal teaching evaluation which each of us got from our freshmen was, "While undoubtedly a brilliant mathematician, Mr. X just can't get it across".

Rickart early on saw the possibilities of an abstract function theory in his papers [53], [54], etc., and through the work of his students. Talking with him and with his student John Holladay got me to thinking about Function Algebras. One day in early 1953 Rickart showed to Kakutani and me a recent paper by the Russian mathematician Leibenson, [45], in which Leibenson raised the following question: Let Γ denote the unit circle $|z| = 1$ and let A denote the disk algebra, viewed as a subalgebra of $C(\Gamma)$. Suppose φ is a function in $C(\Gamma)$ which is not in A. Is the closed algebra generated by φ and A then all of $C(\Gamma)$? He showed that it was if φ is real or if φ satisfies a Lipschitz condition.

Some months earlier I had heard about an intriguing recent result of Rudin: given an algebra of functions continuous in the closed disk $|z| \leq 1$. *Suppose every F in the algebra attains the maximum of its modulus on the boundary $|z| = 1$. Then if one schlicht function belongs to the algebra, every F in the algebra is analytic in $|z| < 1$.*

I did not then know Rudin's proof and spent a week of hard work, making up a proof of Rudin's theorem. My proof was function-algebraic in spirit and rather more complicated than Rudin's own, in [60]. When I saw Leibenson's question, I realized that I could use similar function-algebraic arguments to answer it. I showed that *every closed subalgebra of* $C(\Gamma)$ *which contains* A *either equals* A *or equals* $C(\Gamma)$.

I now asked myself what other closed subalgebras of $C(\Gamma)$ have this property of being "maximal" in $C(\Gamma)$. Let us consider a simple closed curve γ on a Riemann surface which is a torus, such that γ bounds a region Ω on this torus. Then the boundary functions on γ of all functions analytic on Ω and continuous on $\Omega \cup \gamma$ make up such a maximal subalgebra of $C(\gamma)$. Also $C(\gamma) \cong C(\Gamma)$. Since Ω need not be of the type of the disk, we have a new maximal subalgebra of $C(\Gamma)$. It was clear then that one should prove that if Σ is any finite Riemann surface with nice boundary $\gamma = \partial\Sigma$, then the algebra $A(\Sigma)$ of functions analytic on $\Sigma \backslash \partial\Sigma$ and continuous on Σ is a maximal subalgebra of $C(\gamma)$. With the kind help of Maurice Heins at Brown, I proved this for the case that $\partial\Sigma$ is a single contour in [70]. Hal Royden proved the general case in [57].

The algebra considered by Leibenson, generated by φ and A on Γ, evidently is generated by the two functions: φ and z. Let now φ and ψ be any two functions continuous on Γ which together separate points on Γ, and denote by $[\varphi, \psi]$ the closed subalgebra of $C(\Gamma)$ which they generate. Of course, $[\varphi, \psi]$ may equal $C(\Gamma)$, e.g. if $\varphi = \bar{z}$, $\psi = z$. Suppose that $[\varphi, \psi]$ is a proper subalgebra of $C(\Gamma)$. Then we might expect that Γ lies embedded as the boundary curve $\partial\Sigma$ of some finite Riemann surface Σ such that φ and ψ extend analytically from $\partial\Sigma$ to Σ. In that case $[\varphi, \psi]$ would contain only boundary functions of functions analytic on Σ, and hence be a proper subalgebra of $C(\Gamma)$.

I badly wanted to prove that this is what happens. In the case that φ and ψ are real-analytic on Γ and hence can be viewed as defined and analytic in a little annulus containing Γ, I finally did prove it by the end of 1956, in [71].

One can look at this question geometrically, by considering the image X of Γ in $\mathbb{C}^2$ under the map (φ, ψ). Then X is a simple closed curve in $\mathbb{C}^2$ and the hypothesis that $[\varphi, \psi] \neq C(\Gamma)$ is equivalent to the statement that $P(X) \neq C(X)$. The desired conclusion, the existence of a finite Riemann surface in which Γ is embedded, then becomes the existence of a finite Riemann surface Σ in $\mathbb{C}^2$ having X as its boundary.

In this language, and replacing $\mathbb{C}^2$ by $\mathbb{C}^n$, the problem is then as follows. *Given a simple closed curve* X *in* $\mathbb{C}^n$ *with* $P(X) \neq C(X)$. *Show that there exists a finite Riemann surface* Σ *in* $\mathbb{C}^n$ *(possibly admitting singular points) which has* X *as its boundary.* One expects that the finite Riemann surface Σ equals $\widehat{X}$, the polynomially convex hull of X. All this turned out to be true, as long as the curve X has some regularity. I proved it when X is a single real-analytic curve, Stolzenberg did the case when X is the union of finitely many differentiable closed curves [67], and Herbert Alexander [2] did the case when X is merely rectifiable. Bishop's ideas in [14] and [15] played an important role in this work.

One application of this theory of function algebras on the circle came in the work of Royden in [59] on the maximum principle for bounded analytic functions on an open Riemann surface.

Suppose now that we replace the circle Γ by the unit interval I and study the closed subalgebras of $C(I)$ which are uniform algebras on I. The corresponding geometric problem in $\mathbb{C}^n$ is to identify the polynomially convex hull of a Jordan arc in $\mathbb{C}^n$. When J is a regular Jordan arc, satisfying the same smoothness conditions we imposed on the closed curve X above, it turned out that $\hat{J} = J$, i.e., J is polynomially convex. We expect this, since intuitively we feel that "J cannot bound anything". Furthermore, when J is a regular Jordan arc, $P(J) = C(J)$, i.e., every continuous function on J is a uniform limit on J of polynomials in $z_1, \ldots, z_n$. The proof uses both the fact that J is polynomially convex, and that J is smooth, and was given by H. Helson and J. Quigley, in greater generality, in [34].

However, when J is merely topologically a Jordan arc, i.e., homeomorphic to the interval I, J may fail to be polynomially convex. Examples of this were given by me for $n = 3$, [69], and by Rudin for $n = 2$, [61].

The Peak Point Conjecture and Cole's Thesis. As we saw in Section 5 above, Bishop had shown, for an arbitrary compact plane set X, that $R(X) = C(X)$ whenever each point of X is a peak point of $R(X)$. The Peak Point Conjecture was the statement that if A is a uniform algebra on X and if every point of $\mathfrak{M}$ is a peak point of A (in which case, of course, $\mathfrak{M}$ and X coincide), then $A = C(X)$. A related conjecture, due to Gleason, was the statement that $C(X)$ is characterized as a uniform algebra on X by the fact that each part of its maximal ideal space is a single point.

During the 1960s many people tried to settle these conjectures without success. In his remarkable thesis at Yale in 1968, Brian Cole (Rickart's student) disproved both of these conjectures. His procedure was to make repeated adjunction of square roots to a given uniform algebra A so as to end up with an algebra $\tilde{A}$ which is such that every function in $\tilde{A}$ has a square root in $\tilde{A}$. The proof given by Cole may be found in the appendix to A. Browder's book mentioned above in Section 1. Cole's thesis settled a series of other questions as well, and stimulated much further work.

In particular, Richard Basener at Brown was able to modify Cole's construction so as to obtain a compact set X lying on the sphere $|z_1|^2 + |z_2|^2 = 1$ in $\mathbb{C}^2$ such that $R(X)$ provides another counterexample to the peak point conjecture. Here $R(X)$ denotes the uniform closure on X of rational functions in z_1, z_2 which are analytic on X.

Brian Cole joined the Brown department in 1969.

9. Function Algebras on Smooth Manifolds

Let X be a compact set in $\mathbf{C}^n$. Under what conditions on X does $P(X) = C(X)$? Since the maximal ideal space of $C(X)$ is X and the maximal ideal space of $P(X)$ is $\widehat{X}$, a necessary condition for this is that X be polynomially convex.

Suppose now that X is a compact smooth manifold in $\mathbf{C}^n$ with or without boundary, and that X is polynomially convex. Does it follow that $P(X) = C(X)$? As we saw in Section 8, the answer is "Yes" in the case that X is a circle or an arc.

Let k denote the real dimension of X. For $k \geq 2$, it is clear that a new condition enters. If for instance Y is the 2-dimensional disk: $z_1 = \lambda$, $z_2 = \lambda$, $|\lambda| \leq 1$ in $\mathbf{C}^2$, then $\widehat{Y} = Y$ and $P(Y)$ contains exclusively functions analytic on Y. To rule out such a situation, one may consider the tangent space T_x to X for each point x in X. T_x is a k-dimensional real subspace of $\mathbf{C}^n$. If X is a complex-analytic manifold, as in the example, or if merely X contains a complex-analytic submanifold passing through x, this will show up by the presence of a *complex-linear subspace of* $\mathbf{C}^n$ in T_x.

We call such a subspace a *complex tangent* to X at x, and we call the manifold X *totally real* if it has no complex tangents. In 1968–1969 a breakthrough occurred. It was shown, under various conditions of smoothness on X, and arbitrary k, that *if* Σ *is a smooth totally real manifold in* $\mathbf{C}^n$ *and* X *is a compact and polynomially convex subset of* Σ*, then* $P(X) = C(X)$.

The real subspace $\mathbf{R}^n$ of $\mathbf{C}^n$, consisting of all points $(x_1, \ldots, x_n)$ with all x_j real, is evidently a smooth totally real manifold, and every compact subset of $\mathbf{R}^n$ is polynomially convex. So one recovers the Weierstrass approximation theorem.

The above theorem was proved in R. Nirenberg and R. O. Wells, [50], L. Hörmander and J. Wermer, [40], and E. M. Čirka, [24], and the method of proof in these papers was based on Hörmander's solution of the $\overline{\partial}$-problem. Much further work on this problem, with weakened smoothness conditions and simpler, more elementary proofs, was done later on by Weinstock, Berndtsson, Harvey and Wells, and others.

I had earlier, in [73], proved the result for the case $k = 2$ when X is a 2-dimensional smooth disk, and M. Freeman, in [26], had settled the case of general smooth 2-manifolds. The method used by myself and by Freeman depended on the use of the Cauchy transform of a plane measure, and did not generalize to the case $k > 2$.

Bishop disks. Suppose that Σ is a smooth manifold which does have complex tangents. What then? If the dimension k of $\Sigma > n$, elementary linear algebra shows that Σ has complex tangents at every point. In his paper *Differentiable Manifolds in Complex Euclidean Space* in [17], Errett Bishop showed the following: *Assume $k > n$. Fix x in Σ and assume that the dimension of the largest complex-linear subspace of T_x is $k - n$. Then if U is any neighborhood of x on Σ, there exists an analytic disk in $\mathbf{C}^n$ whose boundary lies in U.*

Suppose now that X is a compact set lying on Σ which contains some open subset of Σ. It follows that $\widehat{X}$ contains a multitude of analytic disks whose boundaries lie in X. Every function in $P(X)$ is then analytic on each of these disks.

These "Bishop disks" have turned out to play an important role in the study of analytic continuation in several complex variables.

10. Hanover, N.H., Palo Alto, New Orleans, Yerevan

Many of us got together in the summers at a succession of meetings devoted at least in part to Function Algebras. The atmosphere was rather relaxed some of the time. The conference at Dartmouth College in Hanover was held in 1960. It was organized by Terry Mirkil et al. and was supported by the NSF. One weekday during the meeting Matt Gaffney showed up from Washington, representing the NSF, to see how the conference was going. At the Dartmouth math department he found *none* of the mathematicians, only one of the wives, looking for her husband. The rest of us were out in the lovely countryside. I myself was with Karel de Leeuw and Siggi Helgason on a sailboat on a lake. There were no dire consequences.

In 1961 there was a one-month conference in analysis at Stanford, under the auspices of the American Mathematical Society, and many of us were there and gave talks.

A conference fully devoted to Function Algebras was held at Tulane University in April 1965, organized by Frank Birtel et al. Most people interested in the subject attended, and a volume of the proceedings was published, *Function Algebras*, edited by F. Birtel, Scott-Foresman and Co. (1966).

In September, 1965 a number of us went to a big conference on analytic functions in the Soviet Union in Yerevan. We had a chance to meet and talk with many of the Russians who had similar interests, and I was very much struck by the warmth and friendliness of our hosts. The group around Shabat, Vitushkin, and Mergelyan was very active and doing fundamental work in approximation theory. It included E. Gorin, A. Gončar, E. M. Čirka, S. Melnikov, and E. P. Dolzhenko.

In those days the Russians were not party to an international copyright agreement, so they could freely translate foreign books into Russian. When an author came to the Soviet Union, he got his royalty for the translation in rubles. In this way Hoffman and I got some rubles. They had translated an article of mine so I got enough for one bottle of Armenian cognac and one fur hat. Hoffman's book had been translated, so he was amply supplied with rubles for cognac.

Gončar threw a party for many of us at his family home in Yerevan. The party was very high-spirited with singing, piano-playing and a greater density of liquor bottles on the table than I have ever seen. Of course many toasts were drunk. My toast to Mergelyan was "on your beautiful work which has inspired us all". V. P. Havin responded, with a toast to Mergelyan: "Your work has inspired not only the Americans."

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