DERIVED EQUIVALENCES BETWEEN TORSORS UNDER ABELIAN VARIETIES

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Abstract. We establish a derived equivalence criterion for torsors under abelian varieties over arbitrary fields.

Introduction

In this short note, we generalize the work of Mukai [Mu98, Proposition 4.11] and Gulbrandsen [G08, Proposition 4.1] to obtain a derived equivalence criterion for torsors under abelian varieties:

Theorem. *Let A and B be abelian varieties over a field k. Let X be an A-torsor and Y a B-torsor. Then $D^b(Y) \simeq D^b(X)$ if and only if Y is a fine moduli space of simple, semi-homogeneous sheaves on X with fixed numerical Chern character.*

In Section 1, we establish some basic results on simple semi-homogeneous sheaves and their moduli; in particular, we show that, when they are nonempty, the moduli spaces of simple, semi-homogeneous sheaves with fixed numerical invariants are torsors under abelian varieties. In Section 2, we prove our main theorem. That one may obtain derived equivalences by taking universal sheaves as kernels follows trivially from a weakened version of Bridgeland's equivalence criterion (see [Br99, Theorem 5.1.]) that holds in arbitrary characteristic. To prove the converse direction, we slightly generalize a result of Orlov's, namely [Or02, Proposition 3.2], and show that, up to shift, the kernel of a derived equivalence $\Phi_{\mathcal{E}} : D^b(Y) \xrightarrow{\sim} D^b(X)$ is a Y-flat family of sheaves on X which are simple and semi-homogeneous by Bridgeland's equivalence criterion. The fact that $\mathcal{E}$ is a universal family then follows immediately from the work of Lieblich and Olsson, namely [LO15, Lemma 5.2.]

Conventions and Notation. Unless otherwise specified, we always work over an arbitray field k. Given a morphism $X \to S$, a quasi-coherent sheaf F on X, and a morphism $T \to S$, we let $X_T := X \times_S T$ and we denote F_T the pullback of F along the induced map $X_T \to X$. When $T = \mathrm{Spec(R)}$ for a ring R, we write X_R and F_R, and if $R = k(s)$ for some $s \in S$, we write F_s. We call a quasi-coherent sheaf F on a projective variety X *simple* if for any field extension k'/k, the morphism $k' \to \mathrm{End}(F_{k'})$ is an isomorphism.

All torsors are for the fppf topology. Given an abelian variety A and an A-torsor X, we denote by $\mathcal{P}_X$ (or simply by $\mathcal{P}$) the twisted Poincaré sheaf on $X \times \mathrm{Pic}^0_{X/k}$. When $X = A$, we always assume that $\mathcal{P}$ is normalized. Given an A-torsor X and a morphism $g : T \to \mathrm{Pic}^0_{X/k}$, we denote by $\mathcal{P}_g$ the twisted sheaf $(\mathrm{id} \times g)^*\mathcal{P}_X$ on $X \times T$. Given an A-torsor X and a morphism $f : T \to A$, we denote by $\mathcal{P}_g$ the restriction of the (base change of) the action map $X_T \times_T A_T \to X_T$ to $X_T = X_T \times_T T$ via the map $\mathrm{id} \times_T f$. Given an A-torsor X and a line bundle L on X, we denote by

ϕ_L the morphism $A \to \mathrm{Pic}^0_{X/k}$ induced by the line bundle $\sigma^* L \otimes p_1^* L^{-1}$ on $X \times A$, where $\sigma : X \times A \to X$ denotes the action map.

We denote by $\mathcal{S}pl_{X/k}$ the stack of simple sheaves on a smooth, projective variety X and by $\mathrm{Spl}_{X/k}$ the moduli space of simple sheaves on X, i.e., the rigidification of $\mathcal{S}pl_{X/k}$ along $\mathbb{G}_m$.

Given a smooth, projective variety X, we denote by $\mathrm{CH}_{\mathrm{num}}(X)$ the Chow ring of X modulo numerical equivalence. We write $\mathrm{CH}_{\mathrm{num}}(X)_{\mathbb{Q}}$ for $\mathrm{CH}_{\mathrm{num}}(X) \otimes \mathbb{Q}$. We denote by $D^b(X)$ the bounded derived category of coherent sheaves on X. Given two smooth projective varieties X and Y and an object $E \in D^b(X \times Y)$, we denote by $\Phi_E : D^b(X) \to D^b(Y)$ the Fourier-Mukai transform with kernel E, i.e., the functor $Rp_{2*}(E \otimes^{\mathbb{L}} p_1^*(-))$.

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1. Simple Semi-Homogeneous Sheaves and their Moduli

Let A be a g-dimensional abelian variety over a field k, and let X be an A-torsor over k. Denote by $\sigma : X \times A \to X$ the action of A on X. Using the canonical isomorphism $\hat{A} \simeq \mathrm{Pic}^0_{X/k}$ ([Al02, Theorem 3.0.3], [Ols08, Proposition 2.1.6]), we obtain an action of $A \times \hat{A}$ on $\mathrm{Spl}_{X/k}$ induced by the twisted sheaf

$$(q_{13} \circ (\sigma \times \mathrm{id}))^* \mathcal{F} \otimes p_{13}^* \mathcal{P}_X^{-1}$$

on $X \times A \times \hat{A} \times \mathrm{Spl}_{X/k}$, where p_{ij} denotes the projection onto the ij-th factor from $X \times A \times \hat{A} \times \mathrm{Spl}_{X/k}$, q_{ij} denotes the projection onto the ij-th factor from $X \times \hat{A} \times \mathrm{Spl}_{X/k}$, $\mathcal{F}$ is the universal twisted sheaf on $X \times \mathrm{Spl}_{X/k}$, and $\mathcal{P}_X$ is the twisted Poincaré sheaf on $X \times \hat{A}$. We denote by $\mathbf{S}(F)$ the stabilizer of a simple sheaf F on X, viewed as a point of $\mathrm{Spl}_{X/k}$, for this action. Note that $\mathbf{S}(F)$ is a closed subgroup of $A \times \hat{A}$.

Definition 1.1. A simple sheaf F on X is called *semi-homogeneous* if $\mathbf{S}(F)$ is g-dimensional.

Remark 1.2. In [Mu78], Mukai uses $\Phi(F)$ to denote what we call $\mathbf{S}(F)$. Note that by the remarks made after [Mu78, Definition 3.5], $\Phi(F)$ and $\mathbf{S}(F)$ define the same subgroup of $A \times \hat{A}$.

Proposition 1.3. *Simple semi-homogeneous sheaves admit the following useful description: Let F be a simple sheaf on X. Then,*

$$dim(Ext^1(F, F)) \geq g.$$

Moreover, equality holds if and only if F is semi-homogeneous.

Proof. We may reduce to the case where k is algebraically closed and $X = A$. Our strategy is to reduce to the case where F is locally free, where the result has already been proven ([Mu78, Proposition 3.16, Theorem 5.8]). The fibers of the family $\mathcal{P} \otimes p_1^* F$ of sheaves on A have bounded CM regularity, so by cohomology and base change we obtain that for any sufficiently ample invertible sheaf L on A, $\widehat{F \otimes L} := \Phi_{\mathcal{P}}(F \otimes L)$ is locally free. Moreover,

$$\dim(\mathrm{Ext}^1(\widehat{F \otimes L}, \widehat{F \otimes L})) = \dim(\mathrm{Ext}^1(F \otimes L, F \otimes L)) = \dim(\mathrm{Ext}^1(F, F)). \quad (1)$$

Recall that for any $G \in D^b(A)$ and any $(a, \alpha) \in A(k) \times \hat{A}(k)$

$$\Phi_{\mathcal{P}}(t_a^* G \otimes \mathcal{P}_\alpha^{-1}) \simeq t_\alpha^* \Phi_{\mathcal{P}}(G) \otimes \mathcal{P}_{-a}^{-1}$$

by [Or02, Corollary 2.13]. It follows that

$$\dim(\mathbf{S}(\widehat{F \otimes L})) = \dim(\mathbf{S}(F \otimes L)) = \dim(\mathbf{S}(F)). \tag{2}$$

(1) and (2) allow us to apply [Mu78, Proposition 3.16, Theorem 5.8] to obtain the desired result. □

Using Proposition 1.3, we obtain that the stack $\mathcal{SSH}_{X/k}$ of simple semi-homogeneous sheaves in an open substack of the stack of simple sheaves. We are mainly interested in the substacks $\mathcal{SSH}^\xi_{X/k}$ parameterizing families of simple semi-homogeneous sheaves whose geometric fibers have Chern character equal to (the pullback of) $\xi \in \mathrm{CH}_{\mathrm{num}}(X_{\overline{k}})_{\mathbb{Q}}$. When they are nonempty, these stacks are $\mathbb{G}_m$-gerbes over algebraic spaces which we denote by $SSH^\xi_{X/k}$.

For the remainder of this section, we fix some $\xi \in \mathrm{CH}_{\mathrm{num}}(X_{\overline{k}})_{\mathbb{Q}}$ such that $\mathcal{SSH}^\xi_{X/k}(\overline{k})$ is nonempty. It is easy to see that $A \times \hat{A}$ acts on $SSH^\xi_{X/k}$ by restriction; we will show that this action makes $SSH^\xi_{X/k}$ into a torsor under an abelian variety.

Definition 1.4. Let $A \times \hat{A}$ act on $SSH^\xi_{X/k}$ as above. We define

$$\mathbf{S}(\xi) := \mathrm{Ker}\left((A \times \hat{A}) \to \mathrm{Aut}(SSH^\xi_{X/k})\right)$$

and

$$A(\xi) := (A \times \hat{A})/\mathbf{S}(\xi).$$

Proposition 1.5. *$SSH^\xi_{X/k}$ is a torsor under $A(\xi)$.*

Before giving the proof, we will need two preliminary lemmas.

Lemma 1.6. *Let A be an abelian variety over an algebraically closed field k. Let F and G be simple, semi-homogeneous sheaves on A with $ch(F) = ch(G) \in CH_{num}(A)_{\mathbb{Q}}$. Then there exists $(a, \alpha) \in A(k) \times \hat{A}(k)$ such that*

$$G \simeq t_a^* F \otimes \mathcal{P}_\alpha^{-1}.$$

Proof. When F and G are locally free, this is [Mu78, Theorem 7.11(2)]. We will reduce to this case using the Fourier transform. First note that we may replace F and G by $F \otimes L$ and $G \otimes L$ for any line bundle L. Choosing L sufficiently ample, we may assume that $\Phi_{\mathcal{P}}(F)$ and $\Phi_{\mathcal{P}}(G)$ are locally free. By [Mu78, Theorem 7.11(2)], there exists some $(a, \alpha) \in A(k) \times \hat{A}(k)$ such that

$$\Phi_{\mathcal{P}}(G) = t_{-\alpha}^* \Phi_{\mathcal{P}}(F) \otimes \mathcal{P}_{-a}^{-1}.$$

By [Or02, Corollary 2.13], we obtain that

$$\Phi_{\mathcal{P}}(G) = t_\alpha^* \Phi_{\mathcal{P}}(F) \otimes \mathcal{P}_{-a}^{-1} = \Phi_{\mathcal{P}}\left(t_a^* F \otimes \mathcal{P}_\alpha^{-1}\right).$$

It remains to apply $\Phi_{\mathcal{P}}^{-1}$. □

Lemma 1.7. *Let G be a smooth, commutative group scheme of finite type over an algebraically closed field k, and let G act on an algebraic space X which is locally of finite type. Let $x \in X(k)$, and suppose that the orbit map*

$$\tau_x : G \mapsto X, \quad g \mapsto xg$$

is surjective on geometric points. Then τ_x induces an isomorphism

$$G/stab(x) \simeq X_{red}.$$

Proof. Since G is reduced, the scheme-theoretic image of τ_x is X_{red}. Denote by $\tau_{x,\mathrm{red}} : G \to X_{\mathrm{red}}$ the induced map. By generic flatness, there exists an open subspace $U \subseteq X_{\mathrm{red}}$ such that the restriction $\tau^{-1}_{x,\mathrm{red}}(U) \to U$ is faithfully flat. Taking the union over the translates of this map, we obtain a faithfully flat morphism

$$\tau_{x,\mathrm{red}}G = \bigcup_{g\in G(k)} \tau^{-1}_{x,\mathrm{red}}(U)\cdot g \to \bigcup_{g\in G(k)} U\cdot g = X_{\mathrm{red}},$$

which necessarily factors through the desired isomorphism. □

Proof of Proposition 1.5. Let $F \in \mathcal{SSH}^{\xi}_{X/k}(\overline{k})$. It suffices to show that the orbit map

$$\tau_f : (A\times \hat{A})_{\overline{k}} \to SSH^{\xi}_{X/k,\overline{k}}, \quad (a,\alpha)\mapsto t_a^*F\otimes \mathcal{P}_\alpha^{-1} \tag{3}$$

is the quotient by $\mathbf{S}(F)$. Indeed, this implies that $\mathbf{S}(F) = \mathbf{S}(\xi)_{\overline{k}}$ so that (3) yields an $A(\xi)_{\overline{k}}$-equivariant isomorphism

$$A(\xi)_{\overline{k}} \simeq SSH^{\xi}_{X/k,\overline{k}}.$$

This will show that $SSH^{\xi}_{X/k}$ is an $A(\xi)$-torsor for the fpqc topology. Since $A(\xi)$ is smooth, $SSH^{\xi}_{X/k}$ must also be; from this it follows that the structure map $SSH^{\xi}_{X/k} \to \mathrm{Spec}(k)$ admits sections étale locally so that $SSH^{\xi}_{X/k}$ is an $A(\xi)$-torsor the for the étale topology and therefore for the fppf topology.

After base change, we may assume that $k = \overline{k}$. Lemma 1.6 shows that τ_F is surjective on geometric points. Lemma 1.7 implies that $SSH^{\xi}_{X/k}$ is a g-dimensional scheme. By Proposition 1.3, we see that $SSH^{\xi}_{X/k}$ is regular, and therefore reduced. Hence, Lemma 1.7 gives the desired result. □

Corollary 1.8. *It follows from $\mathbf{S}(F) = \mathbf{S}(ch(F))$ for any simple semi-homogeneous sheaf F on X.*

Proof. Immediate. □

To conclude that $SSH^{\xi}_{X/k}$ is a torsor under an abelian variety, it suffices to show that $\mathbf{S}(\xi)$, and therefore $A(\xi)$, is an abelian variety. Before doing so, we recall some basic results on objects satisfying index theorems.

Definition 1.9. Let A be an abelian variety over a field k and let F be a coherent sheaf on A.

(1) We say that F satisfies *weak index theorem* (W.I.T. for short) if if there exists an integer i_0 such that $\Phi_{\mathcal{P}}(F)$ is a complex concentrated in degree i_0. We call i_0 the *index* of F.
(2) We say that F satisfies *index theorem* (I.T. for short) if there exists an integer i_0 such that for any $\alpha\in\hat{A}$,

$$H^i(A_{k(\alpha)}, F\otimes\mathcal{P}_\alpha) = 0 \text{ unless } i = i_0\ .$$

Once again we call i_0 the *index* of F.

Remark 1.10. By cohomology and base change, we see that if F satisfies I.T., then $\Phi_{\mathcal{P}}(F) = E[-i_0]$, where E is a locally free sheaf on $\hat{A}$.

Proposition 1.11. *Let $\xi \in CH_{num}(A_{\overline{k}})_{\mathbb{Q}}$ be such that some point $F \in \mathcal{SSH}^{\xi}_{A/k}(\overline{k})$ satisfies W.I.T. with index i_0. Then there is an isomorphism*

$$\Theta_{\mathcal{P}} : \mathcal{SSH}^{\xi}_{X/k} \to \mathcal{SSH}^{\hat{\xi}}_{\hat{A}/k}, \quad \mathcal{G} \mapsto R^{i_0}p_{2*}(\mathcal{P} \otimes p_1^*\mathcal{G}),$$

where $\hat{\xi} \in CH_{num}(\hat{A}_{\overline{k}})_{\mathbb{Q}}$ is the image of ξ under the cohomological Fourier-Mukai transform

$$\Phi^{CH}_{\mathcal{P}} : CH_{num}(A_{\overline{k}})_{\mathbb{Q}} \to CH_{num}(\hat{A}_{\overline{k}})_{\mathbb{Q}}, \quad \xi \mapsto \hat{\xi} := p_{2*}(p_1^*\xi \cdot ch(\mathcal{P})).$$

Proof. First we note that by Lemma 1.6 and the formula

$$\Phi_{\mathcal{P}}(t_a^*G \otimes \mathcal{P}_{\alpha}^{-1}) \simeq t_{\alpha}^*\Phi_{\mathcal{P}}(G) \otimes \mathcal{P}_{-a}^{-1},$$

where $(a, \alpha) \in A(k) \times \hat{A}(k)$ and $G \in D^b(A)$, any point of $\mathcal{SSH}^{\xi}_{A/k}$ satisfies W.I.T. with index i_0.

Next we need to check that for any T-point $\mathcal{G} \in \mathcal{SSH}^{\xi}_{A/k}(T)$, $\Theta_{\mathcal{P}}(\mathcal{G})$ is a flat family of simple, semi-homogeneous sheaves on $\hat{A}$ whose geometric fibers have Chern character $\hat{\xi}$. The only part which is not clear is the flatness of $\Theta_{\mathcal{P}}(\mathcal{G})$. It suffices to verify this condition after base change, so we may assume that k is algebraically closed and apply [Mu87, Theorem 1.6(2)(a,b)]

It suffices to verify that $\Theta_{\mathcal{P}}$ is an isomorphism under the assumption that k is algebraically closed. By [Mu87, Theorem 1.1, Proposition 1.5, Theorem 1.6(2)(a)], we see that the morphism

$$\mathcal{SSH}^{\hat{\xi}}_{\hat{A}/\hat{k}} \to \mathcal{SSH}^{\xi}_{A/k}, \quad \mathcal{G} \mapsto (-1_A)^*R^{g-i_0}p_{1*}(\mathcal{P} \otimes p_2^*\mathcal{G})$$

is an inverse to $\Theta_{\mathcal{P}}$. □

Lemma 1.12. *Let A be an abelian variety over an algebraically closed field k and let L be a line bundle on A. Then there is an isomorphism of stacks $\mathcal{SSH}^{\xi}_{A/k} \to \mathcal{SSH}^{\xi \cdot ch(L)}_{A/k}$ which induces an isomorphism $SSH^{\xi}_{A/k} \to SSH^{\xi \cdot ch(L)}_{A/k}$ that is equivariant with respect to*

$$\begin{pmatrix} 1 & 0 \\ \phi_L & 1 \end{pmatrix} : A \times \hat{A} \to A \times \hat{A}.$$

In particular, there is an isomorphism $\mathbf{S}(\xi) \to \mathbf{S}(\xi \cdot ch(L))$.

Proof. This follows from the fact that for any $(f, g) \in A(T) \times B(T)$ and any sheaf F on $A \times T$,

$$t_f^*(F \otimes L_T) \otimes \mathcal{P}_g^{-1} \simeq t_f^*F \otimes L_T \otimes \mathcal{P}^{-1}_{g-\phi_L(f)}.$$

□

Proposition 1.13. *$\mathbf{S}(\xi)$, and therefore $A(\xi)$, is an abelian variety.*

Proof. We may assume that k is algebraically closed and $X = A$. We will reduce to the case where $SSH^{\xi}_{A/k}$ is a moduli space of locally free sheaves, in which case the result is due to Mukai ([Mu78, Theorem 7.1.]).

By Proposition 1.5, $SSH^{\xi}_{A/k}$ is of finite type, so we may choose a sufficiently ample line bundle L on A such that for any point $F \in \mathcal{SSH}^{\xi}_{A/k}(\overline{k})$, $F \otimes L$ satisfies I.T. Applying Proposition 1.11, we obtain from the isomorphism $\Theta_{\mathcal{P}}$ an isomorphism

$$\theta_{\mathcal{P}} : SSH^{\xi \cdot \mathrm{ch}(L)}_{A/k} \to SSH^{\widehat{\xi \cdot \mathrm{ch}(L)}}_{\hat{A}/k}.$$

We claim that $\theta_{\mathcal{P}}$ is equivariant with respect to the isomorphism

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : A \times \hat{A} \to \hat{A} \times A.$$

By [Or02, Corollary 2.13], this is true at the level of closed points, which are dense in $SSH^{\xi \cdot \mathrm{ch}(L)}_{A/k}$. By [Mu78, Proposition 6.16], simple, semi-homogeneous vector bundles are Gieseker stable, so $SSH^{\widehat{\xi \cdot \mathrm{ch}(L)}}_{\hat{A}/k}$ is separated. By Proposition 1.5, $SSH^{\xi \cdot \mathrm{ch}(L)}_{A/k}$ is reduced. It follows that $\theta_{\mathcal{P}}$ is equivariant. Using this along with Lemma 1.12, we obtain isomorphisms

$$\mathbf{S}(\xi) \simeq \mathbf{S}(\xi \cdot \mathrm{ch}(L)) \simeq \mathbf{S}(\widehat{\xi \cdot \mathrm{ch}(L)}).$$

By [Mu78, Theorem 7.1.], $\mathbf{S}(\widehat{\xi \cdot \mathrm{ch}(L)})$ is an abelian variety. □

Corollary 1.14. *$SSH^{\xi}_{X/k}$ is a torsor under an abelian variety.*

Proof. This follows immediately from Proposition 1.5 and Proposition 1.13 □

2. Derived Equivalences

We begin with some generalities.

Lemma 2.1. *Let X, Y be two smooth projective varieties over a field k. Let $\mathcal{E} \in D^b(X \times Y)$. $\Phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$ is an equivalence if and only if $\Phi_{\overline{\mathcal{E}}}$ is.*

Proof. This is [Or02, Lemma 2.12]. □

Proposition 2.2. *Let X, Y smooth projective varieties over an algebraically closed field k. A functor $F : D^b(X) \to D^b(Y)$ is fully faithful if, and only if,*

(1) for each closed point $x \in X$,

$$Ext^0(F\mathcal{O}_x, F\mathcal{O}_x) = k,$$

(2) for each point $x \in X$

$$F : Ext^1(\mathcal{O}_x, \mathcal{O}_x) \to Ext^1(F\mathcal{O}_x, F\mathcal{O}_x)$$

is injective,

(3) for each pair of closed points $x_1, x_2 \in X$ and each $i \in \mathbb{Z}$,

$$Ext^i(F\mathcal{O}_{x_1}, F\mathcal{O}_{x_2}) = 0,$$

unless $x_1 = x_2$ and $0 \leq i \leq dim(X)$.

Proof. Follow the proof of [Br99, Theorem 5.1] and note that the results cited do not use any assumptions on the characteristic. □

Lemma 2.3. *Let X, Y smooth projective varieties over an algebraically closed field k, and let $\mathcal{E}$ be a sheaf on $X \times Y$, flat over X. Then for any closed point $x \in X$, the map $\Phi_{\mathcal{E}} : Ext^1(\mathcal{O}_x, \mathcal{O}_x) \to Ext^1(\mathcal{E}_x, \mathcal{E}_x)$ is the Kodaira-Spencer map for the family $\mathcal{E}$. More precisely, it is defined as follows: identifying $Ext^1(\mathcal{O}_x, \mathcal{O}_x)$ with T_xX, this map is given by sending a morphsim $D := Spec(k[\epsilon]/(\epsilon^2)) \to X$ to $\mathcal{E}\mid_{D\times Y}$, viewed as an element of $Ext^1(\mathcal{E}_x, \mathcal{E}_x)$.*

Proof. This is [H06, Examples 5.4, vii)]. □

Using these, we obtain our main result:

Theorem 2.4. *Let A and B be g-dimensional abelian varieties over a field k, let X be an A-torsor, and let Y be a B-torsor. Then $D^b(Y) \simeq D^b(X)$ if and only if there is some $\xi \in CH_{num}(X_{\overline{k}}) \otimes \mathbb{Q}$ such that $SSH^{\xi}_{X/k}$ is a fine moduli space, isomoporphic to Y.*

Proof. First suppose that $SSH^{\xi}_{X/k}$ is a fine moduli space of semi-homogeneous sheaves on X with universal sheaf $\mathcal{E}$ on $SSH^{\xi}_{X/k} \times X$. We claim that the functor

$$\Phi_{\mathcal{E}} : D^b(SSH^{\xi}_{X/k}) \to D^b(X)$$

is an equivalence. First, we note that by Proposition 1.13, $SSH^{\xi}_{X/k}$ is a g-dimensional smooth, projective variety. By Lemma 2.1, we may reduce to the case where k is algebraically closed. By [H06, Proposition 7.11], it suffices to show that $\Phi_{\mathcal{E}}$ is fully faithful. Conditions (1) and (3) of Proposition 2.2 are immediately satisfied. By Lemma 2.3, the map in condition (2) is the identity, and hence it is an isomorphism.

Now suppose that $D^b(Y) \simeq D^b(X)$. Then by [Or02, Proposition 3.2], we may assume that this equivalence is given by a Fourier-Mukai transform with sheaf kernel $\mathcal{E}$ on $Y \times X$. We claim that $\mathcal{E}$ is a Y-flat family of simple, semi-homogeneous sheaves on X. It suffices to show this after base change, so we may assume that k is algebraically closed and that $Y = A$ and $X = B$. Applying generic flatness to the family $\mathcal{E}$, we can find a closed point $b \in B$ such that $\Phi_{\mathcal{E}}(\mathcal{O}_b)$ is simple semi-homogeneous sheaf on A. By [Or02, Corollary 2.13], we obtain for any $b' \in B$ a pair $(a, \alpha) \in A \times \hat{A}$ such that

$$\Phi_{\mathcal{P}}(\mathcal{O}_{b'}) = \Phi_{\mathcal{P}}(t^*_{b'-b}(\mathcal{O}_b)) = t^*_a\Phi_{\mathcal{P}}(\mathcal{O}_b) \otimes \mathcal{P}_{\alpha}.$$

Hence, for any closed point $b' \in B$, $\Phi_{\mathcal{P}}(\mathcal{O}_{b'})$ is a simple, semi-homogeneous sheaf on B. By [H06, Lemma 3.31], we see that $\mathcal{E}$ is a flat family of simple semi-homogeneous sheaves.

By [LO15, Lemma 5.2], $\mathcal{E}$ induces an open immersion

$$\iota_{\mathcal{E}} : Y \hookrightarrow sD_{X/k},$$

where $sD_{X/k}$ is the moduli space of simple, perfect complexes on X. By the above discussion, $\iota_{\mathcal{E}}$ factors through an open immersion

$$Y \hookrightarrow SSH^{\xi}_{X/k}$$

for some $\xi \in \mathrm{CH}_{\mathrm{num}}(X_{\overline{k}})_{\mathbb{Q}}$. Any open immersion between torsors under abelian varieties is an isomorphism. Hence, this map is an isomorphism and $\mathcal{E}$ is the universal sheaf.

□

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