Hardy Space and Hartogs Triangle

by

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Nomenclature

$\hat{f}_k, \hat{f}(k)$	Fourier coefficient
$\hat{f}(j,k)$	2D Fourier coefficient
$\mathbb{D}$	Open unit disk
$\mathbb{T}$	Unit circle
dm	Normalized Lebesgue measure on $\mathbb{T}$
$L^p(\mathbb{T})$	Space of $p-$integrable functions defined on $\mathbb{T}$
$\mathrm{Hol}(\mathbb{D})$	Space of holomorphic functions defined on $\mathbb{D}$
$\mathbb{H}^p(\mathbb{D})$	Hardy space of the disk
$H^p = H^p(\mathbb{T})$	Hardy space of the circle
$C(\mathbb{T})$	Continuous functions defined on $\mathbb{T}$
z	Identity function on the circle
z^n	Pointwise multiplication of n identity functions
$\mathrm{span}(z^n)_{n\in\mathbb{Z}}$	Polynomials
bf	Boundary limits of f
$\lim_{z\to e^{i\theta}\sphericalangle}$	Nontangetial convergence
S_ξ	Stolz angle
H	Hilbert transform
P_+	Riesz Projection
$L^{p,\infty} = L^{p,\infty}(\mathbb{T})$	Weak L^p space
Log	Complex Logarithm defined on $\mathbb{C} - \{(-\infty, 0)\}$
H^∞_-	L^∞ functions with vanishing nonnegative Fourier coefficients
$C(\mathbb{T})_-$	Continuous functions with vanishing nonnegative Fourier coefficients
$\mathbb{H}$	Hartogs triangle
$d_b(\mathbb{H})$	Distinguished Boundary of the Hartogs triangle
$H^p(\mathbb{H})$	Hardy Space of $\mathbb{H}$
$H^p(d_b(\mathbb{H}))$	Hardy Space of $d_b(\mathbb{H})$
$L^p(d_b(\mathbb{H}))$	Space of p-integrable functions defined on $d_b(\mathbb{H})$
S	Szegő projection
C_0	Functions vanishing at infinity
C^k	Functions with k continuous derivatives
$\mathcal{F}$	Fourier transform on $\mathbb{R}^2$
$\mathcal{S}(\mathbb{R})$	Class of Schwartz functions
Φ_n	n-th Fejer Kernel

Contents

Introduction

Hardy spaces arise naturally when we study the boundary limit of a holomorphic function f defined in the open unit disk $\mathbb{D}$. This is because f has the following power series representation for all $z \in \mathbb{D}$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Formally and heuristically, we can replace z with $e^{i\theta}$ to obtain a function bf defined on $\mathbb{T}$:

$$bf(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}.$$

If this infinite sum is well-defined, then we see that a_n is not only the Taylor coefficient in the power series expansion, but also the Fourier coefficient in the Fourier series expansion. We can regard bf as a function with vanishing negative Fourier coefficients. bf and f can then be identified. To be more precise, the map $f \mapsto bf$ is an isometry between some Banach spaces we will later define. Moreover, we will discuss the following two types of convergence in section 1:

- $\lim_{r \nearrow 1} f(re^{i\theta}) = bf(e^{i\theta})$, which is the convergence in the space L^p equipped with either the norm topology or the weak* topology.
- $\lim_{z \to e^{i\theta} \sphericalangle} f(z) = bf(e^{i\theta})$, which is called the nontangential a.e. convergence.

We will also prove some Banach space properties of H^p. In section 2, we will study two important operators related to the Hardy space: the Hilbert transform H and the Riesz projection P_+. These two operators can be easily defined on the set of polynomials $\text{span}(z^n)_{n \geq 0} \subset L^p(\mathbb{T})$. The natural question is whether they have a continuous extension to the L^p space. Our main work in section 2 is to prove an estimate of their operator norms on different function spaces. Alongside, we will prove that for $f \in L^p$ where $1 < p < \infty$,

$$\sum_{k=-m}^{m} \hat{f}(k) z^k \longrightarrow f,$$

where the convergence is taken in L^p. We will also apply the boundedness of P_+ to prove some duality results of Hardy spaces.

In (1), Monguzzi defines the Hardy space on the Hartogs triangle. The corresponding projection map S is called the Szegő projection. The notion of a Hardy space can be readily generalized to domains with smooth boundaries. However, extending this definition to arbitrary domains remains a non-trivial challenge. The Hartogs triangle is well-known for its pathological behaviors in multivariable complex analysis, which motivates the need for a suitable definition tailored to this specific domain. In section 3, we will generalize some of the classical results of the Hardy space on the disk to the Hardy space on the Hartogs triangle, including the boundedness of S and the duality of Hardy spaces.

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1 Preliminaries - Hardy Space on the Unit Disk

Definition 1.1. Let f be an integrable function defined on the unit disk with respect to the normalized Lebesgue measure. We denote the $k-$th Fourier coefficient by $\hat{f}_k$ or $\hat{f}(k)$, which is equal to

$$\int_{\mathbb{T}} f\bar{z}^k dm.$$

Definition 1.2. Let $\mathbb{D}$ be the open unit disk of the complex plane. Denote its closure by $\bar{\mathbb{D}}$ and its boundary by $\mathbb{T}$. Denote the normalized Lebesgue measure defined on $\mathbb{T}$ by dm. Given a function f defined over $\mathbb{D}$, we can define a function f_r on $\mathbb{T}$ for each $0 \leq r < 1$ by

$$f_r(z) := f(rz).$$

We define the following function spaces:

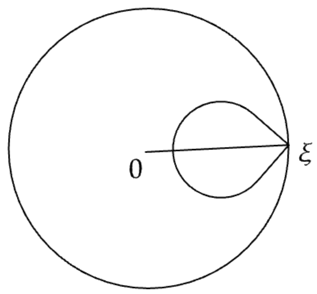
- For $p \in [1, \infty]$, $L^p = L^p(\mathbb{T})$ is the space $p-$integrable functions defined on $\mathbb{T}$.
- $\text{Hol}(\mathbb{D})$ is the space of holomorphic functions defined on the open unit disk.
- For $p \in [1, \infty)$, $\mathbb{H}^p(\mathbb{D}) = \Big\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{\mathbb{H}^p} := \big(\sup_{0 \leq r < 1} \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} \big)^{\frac{1}{p}} < \infty \Big\}$.
- $\mathbb{H}^\infty(\mathbb{D}) = \Big\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{\mathbb{H}^\infty} = \sup_{z \in \mathbb{D}} |f(z)| < \infty \}$.
- For $p \in [1, \infty]$, $H^p = \{ f \in L^p(\mathbb{T}) : \hat{f}(k) = 0, k < 0 \}$, equipped with the usual L^p norm.
- $C(\mathbb{T})$ is the set of continuous functions defined on $\mathbb{T}$.
- $\text{span}(z^n)_{n \geq 0}$ is the set of all finite linear combinations of z^n (i.e. the set of all trignometric polynomials).

$\mathbb{H}^p(\mathbb{D})$ and H^p are both called Hardy spaces because there's an isometry between them.

Theorem 1.1 (Norm and Weak* Convergence)**.** *Let $1 \leq p \leq \infty$.*

- *When $p < \infty$, for each $f \in \mathbb{H}^p(\mathbb{D})$, the limit $\lim_{r \nearrow 1} f_r$ exists in the L^p norm and is denoted by bf.*
- *When $p = \infty$, the limit $\lim_{r \nearrow 1} f_r = bf$ exists in the weak* sense.*
- *For all $1 \leq p \leq \infty$, $f \mapsto bf$ is an isometry from $\mathbb{H}^p(\mathbb{D})$ to H^p.*

Theorem 1.2 (Nontangential Convergence)**.** *For each ξ on $\mathbb{T}$, a Stolz angle S_ξ is an angular sector inside $\mathbb{D}$ centered at ξ such that the straight line connecting 0 and ξ is its bisector and its opening angle is less than π.*



Suppose $f \in \mathbb{H}^p(\mathbb{D})$ for $1 \leq p \leq \infty$. Then the nontangential limit always exists. More precisely, there exists an L^p function bf such that for a.e. ξ on $\mathbb{T}$,

$$\lim_{z \to \xi, z \in S_\xi} f(z) = bf(\xi)$$

We also denote this limit by $\lim_{z \to e^{i\theta} \triangleleft} f(z)$.

Remark. The main tool used in Theorem 1.1 is the weak or weak* compactness of the closed unit ball of L^p spaces, while Theorem 1.2 is proved by using a maximal function. For the details of the proof of Theorem 1.1 and 1.2, see (2). According to this theorem, H^p functions are the boundary values of $\mathbb{H}^p(\mathbb{D})$ functions and $\mathbb{H}^p(\mathbb{D})$ functions are the Poisson integrals of H^p functions. From now on, we use f to denote both a function in $\mathbb{H}^p(\mathbb{D})$ and its boundary limit in H^p.

Theorem 1.3. *$\mathbb{H}^p(\mathbb{D})$ is a Banach space for $p \in [1, \infty]$.*

Proof. We mostly follow Rudin's proof (3). Suppose $(f_n)_{n \geq 0}$ is a Cauchy sequence in $\mathbb{H}^p(\mathbb{D})$. For any $0 \leq r < R < 1$ and any z such that $|z| \leq r$, we apply Cauchy's formula to the function $f_n - f_m$ around the circle γ of radius R to get

$$\begin{aligned}(R - r)|f_n(z) - f_m(z)| &= (R - r)\frac{1}{2\pi}\left|\int_\gamma \frac{f_n(\zeta) - f_m(\zeta)}{\zeta - z} d\zeta\right| \\ &\leq (R - r)\frac{1}{2\pi}\int_0^{2\pi} \frac{|f_n(Re^{i\theta}) - f_m(Re^{i\theta})|}{|Re^{i\theta} - z|} R d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |(f_n - f_m)_R(e^{i\theta})| d\theta,\end{aligned}$$

where the last inequality follows from $R < 1$ and $\frac{R-r}{|Re^{i\theta}-z|} \leq 1$. We then can further deduce that

$$\begin{aligned}(R - r)|f_n(z) - f_m(z)| &\leq \|(f_n - f_m)_R\|_{L^1} \\ &\leq \|(f_n - f_m)_R\|_{L^p} \\ &\leq \|f_n - f_m\|_{\mathbb{H}^p}.\end{aligned}$$

The last expression goes to 0 as $n, m \to \infty$. This implies that the sequence $(f_n)_{n \geq 0}$ is uniformly Cauchy on closed disks contained inside the unit disk. Therefore, it converges uniformly on compact subsets of $\mathbb{D}$ to a function $f \in \mathrm{Hol}(\mathbb{D})$. We need to show that f_n converges to f in the $\mathbb{H}^p(\mathbb{D})$ norm. For every $\varepsilon > 0$, we can find some n so that for every $m_1, m_2 \geq n$, $\|f_{m_1} - f_{m_2}\|_{\mathbb{H}^p} < \varepsilon$. For every $r < 1$, we can use Fatou's lemma to conclude that for every $m_1 \geq n$, we have

$$\begin{aligned}\|(f_{m_1} - f)_r\|_{L^p} &\leq \liminf_m \|(f_{m_1} - f_m)_r\|_{L^p} \\ &\leq \liminf_m \|f_{m_1} - f_m\|_{\mathbb{H}^p} \\ &\leq \varepsilon.\end{aligned}$$

Thus, $\sup_r \|(f_{m_1} - f)_r\|_{L^p} \leq \varepsilon$ and $\|f_{m_1} - f\|_{\mathbb{H}^p} \leq \varepsilon$.

□

Theorem 1.4. *H^p is a Banach space for $p \in [1, \infty]$.*

Proof. By Theorem 1.1, there is an surjective isometry from $\mathbb{H}^p(\mathbb{D})$ to H^p. Since isometry preserves completeness and $\mathbb{H}^p(\mathbb{D})$ is complete by Theorem 1.3, H^p is complete.

□

Lastly, we will prove some density results that are useful for later sections.

Theorem 1.5. $\text{span}(z^n)_{n\geq 0}$ *is dense in* H^p *for* $1 \leq p < \infty$ *and* $H^\infty \cap C(\mathbb{T})$.

Proof. Let Φ_n be the Fejér kernel. It is well-known that for all $p \in [1, \infty)$ and $f \in L^p$,

$$f * \Phi_n(z) = \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} \hat{f}(k) z^k \xrightarrow{L^p} f.$$

That is, the n^{th} arithmetic means of the partial Fourier sum of f converges in L^p to f. The same is true for $p = \infty$ if $f \in C(\mathbb{T})$. If $f \in H^p$ or $H^\infty \cap C(\mathbb{T})$, then by definition it has vanishing negative Fourier coefficients. We can thus deduce that

$$f * \Phi_n(z) = \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=0}^{m} \hat{f}(k) z^k \xrightarrow{L^p} f.$$

This implies the density of $\text{span}(z^n)_{n\geq 0}$ in H^p and $H^\infty \cap C(\mathbb{T})$.

□

Corollary 1.5.1. H^p *for* $1 \leq p < \infty$ *and* $H^\infty \cap C(\mathbb{T})$ *are separable.*

Proof. Consider the following subset of functions:

$$E := \{f : \mathbb{T} \to \mathbb{C} : f(z) = \sum_{k=0}^{n} (a_k + ib_k) z^k, n \in \mathbb{N}, a_k \in \mathbb{Q}, b_k \in \mathbb{Q}\}.$$

It is easy to see that any polynomials can be approximated by polynomials in E.

□

The following corollary is the analogue of the density of L^2 in L^p.

Corollary 1.5.2. H^2 *is dense in* H^p *for* $1 \leq p < \infty$.

Proof. H^2 certainly contains $\text{span}(z^n)_{n\geq 0}$.

□

It is natural to ask whether H^∞ is separable. One should expect that this is not true since L^∞ is not separable. We will follow the hint in (2) to prove this statement.

Theorem 1.6. H^∞ *is not separable.*

Proof. We will construct an uncountable collection of functions $(\Theta_\alpha)_{\alpha \in \mathbb{R}_+} \subset H^\infty$ such that $\|\Theta_\alpha - \Theta_\beta\|_{L^\infty} \geq 1$ whenever $\alpha \neq \beta$. This creates uncountably many disjoint open sets in H^∞, which implies that H^∞ is not separable.

Define

$$\Theta_\alpha(e^{i\theta}) := e^{-\alpha \frac{1+e^{i\theta}}{1-e^{i\theta}}} = \lim_{r \nearrow 1} e^{-\alpha \frac{1+re^{i\theta}}{1-re^{i\theta}}}.$$

We first show that $\Theta_\alpha \in H^\infty$. Clearly Θ_α is the radial limit of the function $e^{-\alpha\frac{1+z}{1-z}} \in \mathrm{Hol}(\mathbb{D})$. By Theorem 1.2, it suffices to show that $e^{-\alpha\frac{1+z}{1-z}} \in \mathbb{H}^\infty(\mathbb{D})$. For all $0 \le r < 1$,

$$1 - 2r\cos\theta + r^2 \ge 1 - 2r + r^2 = (1-r)^2.$$
$$\frac{1-r^2}{1-2r\cos\theta + r^2} \ge 0.$$
$$|e^{-\alpha\frac{1+re^{i\theta}}{1-re^{i\theta}}}| = e^{-\alpha\Re(\frac{1+re^{i\theta}}{1-re^{i\theta}})} = e^{-\alpha\frac{1-r^2}{1-2r\cos\theta+r^2}} \le e^0 = 1.$$

Hence, $\|e^{-\alpha\frac{1+z}{1-z}}\|_{\mathbb{H}^\infty} \le 1 < \infty$. Moreover, the calculation above also suggests that when $r = 1$ and $e^{i\theta} \ne 1$,

$$|\Theta_\alpha(e^{i\theta})| = 1$$

Now we show that $\|\Theta_\alpha - \Theta_\beta\|_{L^\infty} \ge 1$ whenever $\alpha \ne \beta$. We first notice that

$$\|\Theta_\alpha - \Theta_\beta\|_{L^\infty} = \|\Theta_\alpha - \frac{\Theta_\beta}{\Theta_\alpha}\Theta_\alpha\|_{L^\infty} = \||\Theta_\alpha||1-\Theta_{\beta-\alpha}|\|_{L^\infty} = \|1-\Theta_{\beta-\alpha}\|_{L^\infty}$$

Hence, it suffices to show that $\|1-\Theta_\alpha\|_{L^\infty} \ge 1$ for $\alpha > 0$. When $e^{i\theta} \ne 1$, we have

$$\Re(\frac{1+e^{i\theta}}{1-e^{i\theta}}) = \frac{1-|e^{i\theta}|^2}{1-2|e^{i\theta}|\cos\theta + |e^{i\theta}|^2} = 0,$$
$$\Im(\frac{1+e^{i\theta}}{1-e^{i\theta}}) = \frac{\sin\theta}{1-\cos\theta}.$$

By L'Hôpital's rule,

$$\lim_{\theta\searrow 0}\frac{\sin(\theta)}{1-\cos\theta} = \lim_{\theta\searrow 0}\frac{\cos\theta}{\sin\theta} = \infty,$$
$$\lim_{\theta\nearrow 2\pi}\frac{\sin(\theta)}{1-\cos\theta} = \lim_{\theta\nearrow 2\pi}\frac{\cos\theta}{\sin\theta} = -\infty.$$

By the intermediate value theorem, for any $\alpha > 0$, we can find some $0 < \theta_0 < 2\pi$ such that

$$\frac{\alpha\sin\theta_0}{1-\cos\theta_0} = \pi,$$
$$\Theta_\alpha(e^{i\theta_0}) = e^{-\alpha i\Im(\frac{1+e^{i\theta_0}}{1-e^{i\theta_0}})} = e^{-i\frac{\alpha\sin\theta_0}{1-\cos\theta_0}} = e^{-i\pi} = -1,$$
$$|1-\Theta_\alpha(e^{i\theta_0})| = 2.$$

By the continuity of Θ_α around $e^{i\theta_0}$, we can find a subset $A \subset \mathbb{T}$ with positive measure such that $|1-\Theta_\alpha(e^{i\theta})| > 1.5$ on A. We conclude that $\|1-\Theta_\alpha\|_{L^\infty} \ge 1$ for $\alpha > 0$.

$\square$

2 Operators Defined on the Hardy Space

2.1 Boundedness of the Hilbert Transform and the Riesz Projection

Definition 2.1. Let $u \in L^2(\mathbb{T})$ be a real-valued function. Then the Hilbert transform $Hu \in L^2(\mathbb{T})$ of u is defined to be the unique function such that

$$\widehat{Hu}(0) = 0,$$
$$u + i(Hu) \in H^2.$$

Remark. It is known that the Hilbert transform satisfies the following estimate:

$$\|Hu\|_{L^2} \leq \|u\|_{L^2}.$$

If u is a complex-valued function, then we can extend H linearly:

$$Hu := H(\Re u) + iH(\Im u).$$

Here are two formulae for Hu:

$$Hu = \frac{1}{i}(P_+u - P_-u) - \frac{1}{i}\hat{u}(0),$$
$$Hu(z) = \int_{\mathbb{T}} \Im(\frac{\zeta + z}{\zeta - z})u(\zeta)dm(\zeta),$$

where P_+ and P_- are defined by the following formula:

$$P_+u = \sum_{k\geq 0} \hat{u}_k z^k,$$
$$P_-u = \sum_{k<0} \hat{u}_k z^k.$$

These formulae hold at least when u is a polynomial.

Theorem 2.1. *The Hilbert transform is anti-self-adjoint on $L^2(\mathbb{T})$.*

Proof. P_+, P_-, and the function $u \mapsto \hat{u}(0)$ are all orthogonal projections on $L^2(\mathbb{T})$, so they are all self-adjoint. We then use the property that inner product is conjugate linear in the second entry to get

$$\begin{aligned}
\langle Hu, v\rangle &= -i\langle P_+u, v\rangle + i\langle P_-u, v\rangle + i\langle \hat{u}(0), v\rangle \\
&= -i\langle u, P_+v\rangle + i\langle u, P_-v\rangle + i\langle u, \hat{v}(0)\rangle \\
&= \langle u, iP_+v\rangle + \langle u, -iP_-v\rangle + \langle u, -i\hat{v}(0)\rangle \\
&= \langle u, i(P_+v - P_-v) - i\hat{v}(0)\rangle \\
&= -\langle u, Hv\rangle.
\end{aligned}$$

□

Remark. Notice that $L^2 \cap L^1 = L^2$ (because the circle is a space with finite measure) is a dense subspace of L^1. Suppose we do know that H sends functions in L^2 to $L^{1,\infty}$ and is in fact a bounded operator (i.e. for all $f \in L^2$, there exists some constant C such that $\|Hf\|_{L^{1,\infty}} \leq C\|f\|_{L^1}$, which is proved in the following theorem). Then, we can extend H to L^1 and view it as an bounded operator from L^1 to $L^{1,\infty}$. Specifically, for any $g \in L^1$, we find a sequence $\{g_n\}_n \in L^2$ such that

$$g_n \xrightarrow{L^1} g,$$

and then define Hg to be the limit of $\{Hg_n\}_n$ in $L^{1,\infty}$. The limit exists because $\{Hg_n\}_n$ is a Cauchy sequence in $L^{1,\infty}$:

$$\|Hg_n - Hg_m\|_{L^{1,\infty}} = \|H(g_n - g_m)\|_{L^{1,\infty}} \le C\|g_n - g_m\|_1 \to 0 \text{ as } n, m \to \infty.$$

Suppose $\{k_n\}_n$ is another sequence in L^2 that converges to g. Then the following sequence also converges to g:

$$k_1, g_1, k_2, g_2, \cdots$$

For the same reason, the image of this sequence under H is again a Cauchy sequence. Moreover, it contains both $\{Hg_n\}_n$ and $\{Hk_n\}_n$ as its subsequences. Therefore, $\{Hg_n\}_n$ and $\{Hk_n\}_n$ must have the same limit. This suggests that the definition of Hg is independent of the choice of an approximating sequence. In this way, we extends H to get a weak-type (1,1) operator on $L^1(\mathbb{T})$. We will often use a similar type of argument implicity for a continuous extension of an operator throughout the entire thesis.

Theorem 2.2. *The Hilbert transform densely defined on $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$ satisfies a weak type $(1, 1)$ estimate.*

Proof. We follow the hint on (2). For every $u \in L^2(\mathbb{T}) \subset L^1(\mathbb{T})$, we define a distribution function:

$$\lambda_{Hu}(t) = m\{\zeta : |Hu(\zeta)| > t\}.$$

We can assume that $u \ge 0$ by decomposition and $\|u\|_1 = \int_{\mathbb{T}} udm = 1$ by normalization. We show that it satisfies the following inequality:

$$\lambda_{Hu}(t) \le \frac{2}{t}, \text{for all } t > 0.$$

Let $f = u + iHu$ so that $f \in H^2$ (and thus $f(0) = \hat{f}(0) = \hat{u}(0) = 1$). Define

$$\phi_t = 1 + \frac{f - t}{f + t}.$$

For a fixed t, ϕ_t is a bounded function on the circle because

$$|\phi_t| \le 1 + \frac{|f - t|}{|f + t|} = 1 + \sqrt{\frac{(u - t)^2 + (Hu)^2}{(u + t)^2 + (Hu)^2}} \le 2.$$

Therefore,

$$I_t(z) := \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \Re\phi_t(\zeta) dm(\zeta)$$

defines an analytic function on $\mathbb{D}$. Notice that

$$\forall z \in \mathbb{D}, \Re I_t(z) = \int_{\mathbb{T}} \Re(\frac{\zeta + z}{\zeta - z}) \Re\phi_t(\zeta) dm(\zeta) = \Re\phi_t(z).$$

Thus, ϕ_t (it has a holomorphic extension to $\mathbb{D}$) and I_t are two analytic functions with the same real part. It follows from the Cauchy Riemann equations that I_t and ϕ_t only differ by an imaginary constant. Since both $I_t(0) = \int_{\mathbb{T}} \Re\phi_t(\zeta) dm(\zeta)$ and $\phi_t(0) = 1 + \frac{1-t}{1+t}$ are real, $I_t \equiv \phi_t$.

Now if $|Hu| > t$, then we have

$$\begin{aligned}
(Hu)^2 &> t^2, \\
u^2 + (Hu)^2 &> t^2, \\
2u^2 + 2(Hu)^2 + 2tu &> u^2 + t^2 + 2tu + (Hu)^2, \\
2u^2 + 2(Hu)^2 + 2tu &> (u + t)^2 + (Hu)^2, \\
\Re\phi_t = \frac{2u^2 + 2(Hu)^2 + 2ut}{(u + t)^2 + (Hu)^2} &> 1.
\end{aligned}$$

By Chebyshev's inequality, we can conclude that

$$\begin{aligned}\lambda_{Hu}(t) \le \lambda_{\Re\phi_t}(1) \le \int_{\mathbb{T}} \Re\phi_t dm = I_t(0) = \phi_t(0)\\ = 1 + \frac{1-t}{1+t} = \frac{1+t+1-t}{1+t} = \frac{2}{1+t} \le \frac{2}{t}\end{aligned}$$

□

Remark. Now we know that H is defined on L^1 (with codomain in $L^{1,\infty}$), so it is defined on L^p for any $1 \le p \le \infty$.

Theorem 2.3. *The Hilbert transform is a strong (p,p) operator any $1 < p < \infty$.*

Proof. The case that $1 < p < 2$ follows immediately from Theorem 2.2, L^2 boundedness of H, and the Marcinkiewicz Interpolation Theorem. If $2 < p < \infty$, then for every $u \in L^p$, we can define a linear functional on $L^2 \subset L^{p'}$ ($\frac{1}{p} + \frac{1}{p'} = 1$):

$$v \in L^2 \mapsto \int_{\mathbb{T}} v\overline{Hu}dm.$$

Since H is anti-self-adjoint, we have:

$$\begin{aligned}|\int_{\mathbb{T}} v\overline{Hu}dm| &= |\int_{\mathbb{T}} (Hv)\bar{u}dm|\\ &\le \|u\|_{L^p}\|Hv\|_{L^{p'}} \text{ (Hölder's inequality)}\\ &\le \|u\|_{L^p}\|H\|_{L^{p'}\to L^{p'}}\|v\|_{L^{p'}}.\end{aligned}$$

This functional can then be extended to $L^{p'}$ by density. The duality between L^p and $L^{p'}$ then implies that $\|Hu\|_{L^p} \le \|H\|_{L^{p'}\to L^{p'}}\|u\|_{L^p}$.

□

Theorem 2.4. *The Riesz Projection P_+ is a strong (p,p) operator for any $1 < p < \infty$.*

Proof. Recall that

$$\begin{aligned}P_+u - P_-u &= iHu + \hat{u}(0),\\ 2P_+u &= iHu + \hat{u}(0) + u.\end{aligned}$$

By Theorem 1.4, we have

$$\begin{aligned}\|P_+u\|_{L^p} &= \frac{1}{2}\|iHu + \hat{u}(0) + u\|_{L^p}\\ &\le \frac{1}{2}(\|H\|_{L^p\to L^p} + 1)\|u\|_{L^p} + \frac{1}{2}|\hat{u}(0)|\\ &\le \frac{1}{2}(\|H\|_{L^p\to L^p} + 1)\|u\|_{L^p} + \frac{1}{2}\|u\|_{L^1}\\ &\le (\frac{1}{2}\|H\|_{L^p\to L^p} + 1)\|u\|_{L^p}.\end{aligned}$$

□

Theorem 2.5. *The Hilbert transform is not bounded on L^∞ and L^1.*

Proof. We first consider the space L^∞. Consider the function $f(z) = i\,\mathrm{Log}(1-z)$ (we use Log to denote the complex logarithm defined on $\mathbb{C} - \{(-\infty, 0]\}$), which is holomorphic in the open unit disk $\mathbb{D}$ and can be extended continuously to $\overline{\mathbb{D}} - \{1\}$. f has the following power series representation in $\mathbb{D}$:

$$f(z) = i\,\mathrm{Log}(1-z) = -\sum_{k=1}^{\infty} \frac{i}{k} z^k.$$

f belongs to $H^2(\mathbb{D})$ because

$$\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{k \ge 0} |\hat{f}(k)|^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

The boundary value bf of f thus belongs to H^2. Since bf is the a.e. nontangential limit of f and f is continuous on $\overline{\mathbb{D}} - \{1\}$, we have

$$i\,\mathrm{Log}(1 - e^{i\theta}) = bf(e^{i\theta}) \in H^2.$$

Notice that the real part $u := \Re bf$ is equal to

$$-\arctan\left(\frac{\sin\theta}{\cos\theta - 1}\right),$$

which is in L^∞ because arctan is a bounded function. The imaginary part $\Im bf$ is unbounded and has mean value zero by the next lemma. Consequently, $Hu = \Im bf$, which shows that H does not even map L^∞ into L^∞.

Now suppose by way of contradiction H is a bounded operator from L^1 to L^1. We can then define a linear functional on L^2 for each $u \in L^\infty$ by the following formula:

$$v \in L^2 \mapsto \int_{\mathbb{T}} v\overline{Hu}\,dm.$$

Again we use Hölder's inequality and the anti-self-adjoint property of H to deduce that

$$\begin{aligned} \left|\int_{\mathbb{T}} v\overline{Hu}\,dm\right| &= \left|\int_{\mathbb{T}} (Hv)\overline{u}\,dm\right| \\ &\le \|u\|_{L^\infty} \|Hv\|_{L^1} \\ &\le \|u\|_{L^\infty} \|H\|_{L^1 \to L^1} \|v\|_{L^1}. \end{aligned}$$

This functional can then be extended to L^1, which then shows that $\|Hu\|_{L^\infty} \le \|H\|_{L^1 \to L^1} \|u\|_{L^\infty}$. This contradicts what we proved above.

□

Lemma 2.6. $\log|1-z|$ *has mean value zero on the circle.*

Proof. Consider the function $h_r(\theta) = \log|1 - re^{i\theta}|$ a.e. defined on the interval $[0, 2\pi]$ for some $r \in [0,1]$. h_r is the restriction of the real part of the function $i\,\mathrm{Log}(1 - rz)$, which is harmonic on a open domain containing $\bar{\mathbb{D}}$ when $r < 1$, by the mean value property of harmonic functions, we have

$$\frac{1}{2\pi} \int_0^{2\pi} h_r(\theta)\,d\theta = 0,$$

$$\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} h_r(\theta)\,d\theta = 0.$$

We are done if we can interchange the order of limit and integral. Hence, our goal now is to justify the use of the dominated convergence theorem. We use Euler's formula to get

$$h_r(\theta) = \log \sqrt{1 + r^2 - 2r\cos\theta}.$$

From this equality, we can see that h_r is symmetric around π, so we just need to focus on the behavior of h_r on $[0, \pi]$. Choose some small $\varepsilon > 0$ so that

- when $1 - r$ is small enough, $|1 - re^{i\theta}| < 1$ for all $\theta \in [0, \varepsilon)$ and
- $\theta^2 < \sin\theta$ for all $\theta \in [0, \varepsilon)$.

Such ε exists by the continuity of the function $|1 - re^{i\theta}|$. Then, $|h_r(\theta)|$ has a uniform bound M over the interval $[\varepsilon, \pi]$ because it can be viewed as a two-variable continuous function defined on the compact space $[0, 1] \times [\varepsilon, \pi]$. We claim that $|h_r(\theta)|$ is dominated by the following function:

$$p(\theta) := \begin{cases} \log(\frac{1}{\theta^2}), & \theta \in [0, \varepsilon) \\ M, & \theta \in [\varepsilon, \pi] \end{cases}.$$

Indeed, we have just shown that $|h_r(\theta)| \leq M$ for $\theta \geq \varepsilon$. Now we fix some $0 < \theta < \varepsilon$. By calculating the derivative of $1 + r^2 - 2r\cos\theta$ with respect to r, one can know that this expression reaches its minimum at $r = \cos\theta$. Hence,

$$\begin{aligned} \theta^2 &\leq |\sin\theta| \\ &= \sqrt{1 + \cos^2\theta - 2\cos^2\theta} \\ &\leq \sqrt{1 + r^2 - 2r\cos\theta} \\ &= |1 - re^{i\theta}| \\ &< 1. \end{aligned}$$

The monotonity of logarithm implies that

$$\log\theta^2 \leq \log|1 - re^{i\theta}| < \log 1 = 0,$$
$$|h_r(\theta)| = |\log|1 - re^{i\theta}|| \leq \log(\frac{1}{\theta^2}).$$

Lastly, we need to show that the function p is integrable. By the monotone convergence theorem and the L'Hôpital's rule, we have

$$\begin{aligned} \int_0^\varepsilon \log(\frac{1}{\theta^2})d\theta &= \lim_{a \searrow 0} \int_a^\varepsilon \log(\frac{1}{\theta^2})d\theta \\ &= \lim_{a \searrow 0} \left[x\log(\frac{1}{x^2}) + 2x\right]_a^\varepsilon \\ &= \varepsilon\log(\frac{1}{\varepsilon^2}) + 2\varepsilon - \lim_{a \searrow 0}(-2a\log a + 2a) \\ &= \varepsilon\log(\frac{1}{\varepsilon^2}) + 2\varepsilon \\ &< \infty. \end{aligned}$$

□

Remark. Since

$$\begin{aligned} iHu &= P_+u - P_-u - \hat{u}(0) \\ &= P_+u - P_-u - \hat{u}(0) + u - u \\ &= 2P_+u - \hat{u}(0) - u, \end{aligned}$$

P_+ cannot be a bounded operator on L^∞ or L^1, as this would imply that H is bounded on these two spaces, leading to a contradiction. We can also directly prove that P_+ is not bounded on L^∞ by considering the following function

$$g(\theta) = \begin{cases} \pi - \theta, & 0 < \theta < \pi \\ -\pi - \theta, & -\pi < \theta < 0 \end{cases}.$$

We can compute its Fourier coefficients. When $k \neq 0$, we have

$$\begin{aligned} \hat{g}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-ik\theta}d\theta \\ &= \frac{1}{2\pi}\int_{0}^{\pi} (\pi - \theta)e^{-ik\theta}d\theta + \frac{1}{2\pi}\int_{-\pi}^{0}(-\pi - \theta)e^{-ik\theta}d\theta \\ &= \frac{1}{2\pi}\left(\frac{-i\pi k + e^{i\pi k} - 1}{k^2} - \frac{i\pi k + e^{-i\pi k} - 1}{k^2}\right) \\ &= \frac{1}{2\pi}\left(\frac{-2\pi i k}{k^2}\right) \\ &= -\frac{i}{k}. \end{aligned}$$

When $k = 0$, $\hat{g}(k)$ is just the integral of g. Since g is an odd function, $\hat{g}(0) = 0$. Hence,

$$P_+g(e^{i\theta}) = -\sum_{k=1}^{\infty} \frac{i}{k}e^{ik\theta} = i\,\mathrm{Log}(1 - e^{i\theta}),$$

where the last equality is a consequence of Abel's theorem. Since $i\,\mathrm{Log}(1 - e^{i\theta})$ is unbounded around $\theta = 0$, we can conclude again that P_+ is unbounded on L^∞.

Theorem 2.7. *The Hilbert transform and the Riesz projection are not bounded on* $C(\mathbb{T})$.

Proof. By using the formula in the remark above, the unboundedness of P_+ follows from the unboundedness of H. Hence, we will just show that H is not bounded.

We prove by contradiction. Suppose $H : C(\mathbb{T}) \to C(\mathbb{T})$ is bounded (or H has a continuous extension to $C(\mathbb{T})$). Let $M(\mathbb{T})$ be the space of Radon measures. The norm of a measure $\nu \in M(\mathbb{T})$, denoted by $|\mu|$, is its total variation. For a fixed $f \in L^1$, $\|f\|_{L^1} = |fdm|$ (see Chapter 3 and 7 in (4)) and defines a bounded linear functional I_f on $C(\mathbb{T})$ by the following formula:

$$\text{for all } g \in C(\mathbb{T}), I_f(g) = \int_{\mathbb{T}} gf dm.$$

By the Riesz representation theorem (see Chapter 7 in (4)), which describes an isometric isomorphism from $M(\mathbb{T})$ to $(C(\mathbb{T}))^*$, the operator norm of I_f is equal to the total variation norm $|fdm|$, which means that

$$\|f\|_{L^1} = \|\bar{f}\|_{L^1} = |\bar{f}dm| = \sup\{|\int_{\mathbb{T}} g\bar{f}dm| : g \in C(\mathbb{T}), \|g\|_{L^\infty} \leq 1\}.$$

We can then use this quantity to estimate the L^1 norm of Hf. Suppose first that $f \in L^2$. For all $g \in C(\mathbb{T})$, we can use the self-adjoint property of H to deduce that

$$\begin{aligned} |\int_{\mathbb{T}} g\overline{Hf}dm| &= |\int_{\mathbb{T}} Hg\bar{f}dm| \\ &\leq \|Hg\|_{L^\infty}\|f\|_{L^1} \\ &\leq \|H\|_{C(\mathbb{T})\to C(\mathbb{T})}\|g\|_{L^\infty}\|f\|_{L^1}. \end{aligned}$$

Hence,

$$\|Hf\|_{L^1} \le \|H\|_{C(\mathbb{T})\to C(\mathbb{T})} \|f\|_{L^1}.$$

As L^2 is dense in L^1, this inequality above is true for any $f \in L^1$. This means that H is continuous from L^1 to L^1, which contradicts Theorem 2.5.

□

2.2 The Nonexistence of Projection onto H^1 and H^∞.

Definition 2.2. Let X be a Banach space and M a closed subspace. A bounded linear operator $P : X \to X$ is called a projection onto M if $P^2 = P$ and its image is equal to M.

Theorem 2.8. *There does not exist any bounded projection from L^1 onto H^1.*

This result was first proved by Newman in (5). Later, Rudin gives another proof in (6). In the following, I will explain Rudin's approach.

Proof. Here is an outline of Rudin's proof. Suppose $P : L^1 \to L^1$ is a bounded projection onto H^1. We can use P and a vector-valued integral that we will later define in this section to obtain another bounded projection P^* onto H^1 that commutes with translation (given $f \in L^1$, its translation by $\zeta \in \mathbb{T}$ is the function $\tau_\zeta f(z) = f(z\bar{\zeta})$). That is, for all $f \in L^1(\mathbb{T})$, $\zeta \in \mathbb{T}$,

$$\tau_\zeta(P^* f) = P^*(\tau_\zeta f).$$

Then by a standard result about the Fourier multiplier operators, see (7), there exists a sequence $\{c_n\} \in l^\infty$ such that for a smooth function f,

$$P^* f(z) = \sum_{n \in \mathbb{Z}} c_n \hat{f}(n) z^n.$$

Apply the equality $P^{*2} = P^*$ to $f(z) = z^n$ for all $n \in \mathbb{Z}$, we get

$$\begin{aligned} P^{*2}(z^n) &= P^*(z^n), \\ P^*(c_n z^n) &= c_n z^n, \\ c_n^2 z^n &= c_n z^n, \\ c_n &= 0 \text{ or } 1, \\ P^*(z^n) &= 0 \text{ or } z^n. \end{aligned}$$

Since $P^* L^1 \subset H^1$ and does not include z^n for $n < 0$, $P^*(z^n) = 0$ for $n < 0$. Since P^* is surjective onto its image H^1, $P^*(z^n) = z^n$ for $n \geq 0$. Therefore, P^* is equal to the Riesz projection P_+ (at least on the set of trignometric polynomials), which is a contradiction because we have already shown that P_+ cannot be extended to a bounded projection on L^1.

□

We see that the essence of the proof is to construct a bounded projection that commutes with translation. Therefore, we introduce some basic knowledge about topological vector space and vector-valued integral. Many definitions and theorems in this subsection can be found in Rudin's functional analysis (8). We only state theorems essential to the construction of P^* without giving a proof.

Definition 2.3. Let X be a topological vector space (TVS) over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We always assume that a TVS satisfies the T_1 axiom and is thus Hausdorff. A local base of X is a collection $\mathcal{B}$ of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathcal{B}$. X is locally convex if there is a local base $\mathcal{B}$ whose members are convex. X is an $F-$space if its topology is induced by a complete, translation invariant metric d. X is a Fréchet space if X is a locally convex $F-$space. The dual of X, denoted by X^*, is the set of all continuous linear functionals on X.

In order to ensure that the integral of a reasonably nice function exists, we need some criterion for compactness.

Definition 2.4. Let $E \subset X$. The convex hull of E is the intersection of all convex subsets of X which contain E, and it is denoted by $\mathrm{co}(E)$. Its closure is denoted by $\overline{\mathrm{co}}(E)$. E is bounded if for every neighborhood V of 0 in X, there exists a number $s > 0$ such that $E \subset tV$ for every $t > s$. E is said to be totally bounded if for every neighborhood V of 0 in X, there exists a finite set F such that $E \subset F + V$.

Theorem 2.9. *If X is locally convex and E is bounded, then $co(E)$ is totally bounded. If X is Fréchet and E is compact, then $\overline{co}(E)$ is compact.*

We also want the integral that we later define to be unique. The following theorem is essential for uniqueness.

Theorem 2.10. *If X is locally convex, then X^* separates points. That is, for every pair of distinct points $x_1, x_2 \in X$, there exists a continuous linear functional Λ such that $\Lambda(x_1) \neq \Lambda(x_2)$. In particular, the dual of a Fréchet space separates points.*

We are ready to define a vector-valued integral. Assume $f : Q \to X$ is a continuous function from a compact measure space Q with a (positive) Borel probability measure μ to a TVS X such that X^* separates points. Since f is continuous, for all linear functional $\Lambda \in X^*$, $\Lambda \circ f$ is continuous and thus measurable. $\Lambda(f(Q))$ is a compact subset of $\mathbb{C}$ and thus bounded, which implies that $\Lambda \circ f$ is an integrable function defined on Q. The following expression is then well-defined:

$$\int_Q (\Lambda \circ f) d\mu.$$

We want the integral to behave nicely with linear functionals. For example, we can always change the order of integration and scalar multiplication for a scalar-valued integral. We include this property into our definition of a vector-valued integral.

Definition 2.5. Let $f : Q \to X$ be a continuous function from a compact measure space Q with a (positive) Borel probability measure μ to a TVS X such that X^* separates points. If there exists a vector $y \in X$ such that for all linear functionals $\Lambda \in X^*$,

$$\Lambda y = \int_Q (\Lambda \circ f) d\mu,$$

then we define the weak integral of f to be y, which is denoted by

$$\int_Q f d\mu.$$

We can apply Theorem 2.9 to see that this weak integral is unique if it exists.

Remark. Notice that if $T : X \to Z$ is a continuous linear operator, then for all $\Lambda \in Z^*$, $\Lambda \circ T$ is again an element of X^* and,

$$\Lambda(Ty) = \int_Q (\Lambda \circ T \circ f) d\mu,$$
$$Ty = \int_Q T f d\mu.$$

T thus commutes with the integral.

Theorem 2.11. *Let $f : Q \to X$ be a continuous function from a compact measure space Q with a (positive) Borel probability measure μ to a TVS X such that X^* separates points. If $\overline{co}(f(Q))$ is compact in X, then the integral of f exists. Moreover, the integral belongs to $\overline{co}(f(Q))$. In particular, the integral of f exists uniquely when X is Fréchet by Theorem 2.9.*

In the case that X is a Banach space, we also have the following triangle inequality for the weak integral:

Theorem 2.12. *Suppose X is Banach. Then by an application of the Hahn-Banach theorem, we have*

$$\| \int_Q f d\mu \| \leq \int_Q \|f\| d\mu.$$

As mentioned in the beginning of this subsection, we will construct a bounded projection $P^* : L^1 \to H^1$ that commutes with translation given any bounded projection $P : L^1 \to H^1$. In fact, we will prove a more general statement.

Theorem 2.13. *Let X be a Banach space and G a compact, abelian topological group acting continuously on X. This means that each $g \in G$ corresponds to a linear continuous isomorphism T_g of X (and we denote $T_g(x)$ by gx) such that*

$$(g, x) \mapsto gx$$

is not only a group action but also a continuous function from $G \times X$ to X. Let $P : X \to X$ be a bounded projection onto a closed, translation-invariant subspace $Y \subset X$ (i.e. for all $g \in G$, $gY \subset Y$). P induces a bounded projection $P^* : X \to X$ onto Y such that P^* commutes with every $g \in G$.

Remark. We claim that the conditions of this theorem are satisfied in the case that $G = \mathbb{T}^n$ and $X = L^1(\mathbb{T}^n)$, which is the space of integrable functions with respect to the normalized Haar measure μ. The group multiplication of $\mathbb{T}^n$ is just pointwise multiplication in each component. The $n-$torus acts on $L^1(\mathbb{T}^n)$ by translation:

$$\begin{aligned} \text{For all } f \in L^1(\mathbb{T}^n), z = (z_1, \cdots, z_n) \in \mathbb{T}^n, \zeta &= (\zeta_1, \cdots, \zeta_n) \in \mathbb{T}^n, \\ \bar{\zeta} &= (\bar{\zeta_1}, \cdots, \bar{\zeta_n}). \\ \tau_\zeta f(z) &= f(z\bar{\zeta}). \end{aligned}$$

We need to show that $(\zeta, f) \mapsto \tau_\zeta f$ is a continuous function from $\mathbb{T}^n \times L^1(\mathbb{T}^n)$ to $L^1(\mathbb{T}^n)$. We apply the triangle inequality to see that

$$\begin{aligned} \|\tau_\zeta f - \tau_{\zeta_0} f_0\|_{L^1} &= \|\tau_\zeta f - \tau_\zeta f_0 + \tau_\zeta f_0 - \tau_{\zeta_0} f_0\|_{L^1} \\ &\leq \|\tau_\zeta f - \tau_\zeta f_0\|_{L^1} + \|\tau_\zeta f_0 - \tau_{\zeta_0} f_0\|_{L^1} \\ &= \|f - f_0\|_{L^1} + \|\tau_\zeta f_0 - \tau_{\zeta_0} f_0\|_{L^1} \end{aligned}$$

where we use the equality $\|\tau_\zeta f - \tau_\zeta f_0\|_{L^1} = \|f - f_0\|_{L^1}$, which follows from the translation invariance of the Haar measure. By definition, $\|f - f_0\|_{L^1}$ goes to 0 as $f \to f_0$. $\|\tau_\zeta f_0 - \tau_{\zeta_0} f_0\|_{L^1}$ goes to 0 because translation is continuous in L^1. Hence, $\tau_\zeta f$ converges to $\tau_{\zeta_0} f_0 \to \zeta$ as $f \to f_0$, $\zeta \to \zeta_0$.

Remark. We show that $H^p \subset L^p$ is a closed, translation invariant subspace for $1 \leq p \leq \infty$.

- H^p is complete and thus a closed subset of L^p.
- If $\hat{f}(n) = 0$, then $\widehat{\tau_\zeta f}(n) = 0$ because

$$\begin{aligned} \widehat{\tau_\zeta f}(n) &= \int_{\mathbb{T}} f(\zeta z) \overline{z}^n dm \\ &= \int_{\mathbb{T}} f(\zeta z) \zeta^n \overline{\zeta z}^n dm \\ &= \zeta^n \hat{f}(n). \end{aligned}$$

Hence, if $f \in H^p$, then $\hat{f}(n) = 0$ for $n < 0$ and thus $\widehat{\tau_\zeta f}(n) = 0$ for $n < 0$, which implies that $\tau_\zeta f \in H^p$. We conclude that H^p is translation invariant.

Proof of Theorem 2.13. We define P^* to be the following vector-valued integral:

$$P^* x := \int_G g^{-1} P g x d\mu(g).$$

μ is the normalized Haar measure. Notice that for a fixed x, $g^{-1}Pgx$ is a continuous function from G to X. Since X is a Banach space and thus a Fréchet space, P^*x is well-defined by Theorem 2.11. P^* is clearly linear. We need to check that

- P^* is bounded. By the next lemma, there exists a uniform bound M such that $\forall g \in G$, $\|g\| \leq M$. We can then use Theorem 2.12 to obtain the following inequality:

$$\begin{aligned} \|P^* x\| &\leq \int_G \|g^{-1} P g x\| dg \\ &\leq \int_G M^2 \|P\| \|x\| dg \\ &= M^2 \|P\| \|x\|. \end{aligned}$$

  The operator norm of P^* is thus bounded by $M^2\|P\|$.

- the image of P^* is Y. $Pgx \in Y$ by the definition of P. Since Y is translation-invariant, $g^{-1}Pgx \in Y$. Let $f(g) = g^{-1}Pgx$. Then Theorem 2.11 tells us that P^*x lies in $\overline{\text{co}}(f(G))$. We just showed that $f(G) \subset Y$. Since Y is both closed and convex, $\overline{\text{co}}(f(G)) \subset Y$ and $P^*x \in Y$.

- P^* is idempotent. It suffices to show that P^* is equal to identity on Y. If $x \in Y$, then $gx \in Y$ and $Pgx = gx$ because P is equal to identity on Y. In this case, $g^{-1}Pgx = g^{-1}gx = x$ is a constant function (independent of g) and thus $P^*x = x$.

- for all $a \in G$, $P^*a = aP^*$. Since μ is translation invariant, the integral of $f(ga)$ is equal to the integral of $f(g)$. Recall that a linear bounded operator always commutes with the integral. In particular, T_a commutes with the integral. We then can conclude that

$$\begin{aligned} P^*(ax) &= \int_G g^{-1} P(gax) d\mu(g) \\ &= \int_G a(ga)^{-1} P(gax) d\mu(g) \\ &= a \int_G (ga)^{-1} P(gax) d\mu(g) \\ &= a \int_G f(ga) d\mu(g) \\ &= a \int_G f(g) d\mu(g) \\ &= aP^* x. \end{aligned}$$

□

Lemma 2.14. $\|g\| := \|T_g\|_{X \to X}$ *is uniformly bounded.*

Proof. Rudin's original proof relies on the Baire category theorem. Here I give an easier argument using the uniform boundedness principle. By this principle, it suffices to show that for all $x \in X$,

$$\sup_{g \in G} \|gx\| < \infty$$

Notice that since x is fixed, the function $g \mapsto \|gx\|$ is continuous. Since G is assumed to be compact, the image of this function is a compact set in $\mathbb{R}$ and is thus bounded.

□

Theorem 2.15. *There does not exists any bounded projection from L^∞ onto H^∞.*

Proof. Again we prove by contradiction. Suppose $P : L^\infty \to L^\infty$ is a bounded projection onto H^∞. $\mathbb{T}$ also acts on $C(\mathbb{T})$ continuously by translation because each $f \in C(\mathbb{T})$ is uniformly continuous and translation always preserves the supremum norm. Hence, for each $f \in C(\mathbb{T})$ and $\zeta \in \mathbb{T}$, the function $\zeta \mapsto \tau_\zeta f$ is continuous from $\mathbb{T}$ to L^∞ (as the norms of L^∞ and $C(\mathbb{T})$ are the same). We can then deduce that $\zeta \mapsto \tau_{\zeta^{-1}}(P(\tau_\zeta f))$ is a continuous function from $\mathbb{T}$ to L^∞, which implies that that the following integral is still well-defined and lies in L^∞:

$$P^* f = \int_{\mathbb{T}} \tau_{\zeta^{-1}}(P(\tau_\zeta f)) dm(\zeta).$$

Moreover, we can use Theorem 2.12 and the fact that translation by a fixed ζ is an isometry to estimate the norm of $P^* f$:

$$\begin{aligned}\|P^* f\|_{L^\infty} &\le \int_{\mathbb{T}} \|\tau_{\zeta^{-1}}(P(\tau_\zeta f))\|_{L^\infty} dm \\ &\le \int_{\mathbb{T}} \|P\|_{L^\infty \to L^\infty} \|f\|_{L^\infty} dm \\ &= \|P\|_{L^\infty \to L^\infty} \|f\|_{L^\infty}.\end{aligned}$$

P^* is thus a bounded operator from $C(\mathbb{T})$ to L^∞. Suppose we can show that $P^* = P_+$ on z^n for each $n \in \mathbb{Z}$. This implies that $P^*(\text{span}(z^n)_{n\in\mathbb{Z}}) \subset \text{span}(z^n)_{n\ge 0}$ and by the continuity of P^* and Theorem 1.5, we have

$$P^*(C(\mathbb{T})) = P^*(\overline{\text{span}(z^n)_{n\in\mathbb{Z}}}) \subset \overline{P^*(\text{span}(z^n)_{n\in\mathbb{Z}})} \subset \overline{\text{span}(z^n)_{n\ge 0}} = H^\infty \cap C(\mathbb{T}) \subset C(\mathbb{T}).$$

P^* is thus a continuous extension of P_+ from $C(\mathbb{T})$ to $C(\mathbb{T})$, which contradicts Theorem 2.7.

Hence, our goal now is to show that for each $n \in \mathbb{Z}$, $P^* z^n = P_+ z^n$. Recall that the weak integral commutes with every bounded linear functional in $(L^\infty)^*$ and in particular, the Fourier coefficients operator Λ^k defined by $\Lambda^k(f) = \hat{f}(k)$ for each $k \in \mathbb{Z}$. Let's do some calculations:

$$\begin{aligned}\Lambda^k(P^* z^n) &= \Lambda^k\Big(\int_{\mathbb{T}} \tau_{\zeta^{-1}}(P(\tau_\zeta z^n)) dm(\zeta)\Big) \\ &= \int_{\mathbb{T}} \Lambda^k(\tau_{\zeta^{-1}}(P(\tau_\zeta z^n))) dm(\zeta) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} (P(z^n \bar{\zeta}^n))(\xi\zeta) \bar{\xi}^k dm(\xi) dm(\zeta) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} (P(z^n \bar{\zeta}^n))(\xi) \bar{\xi}^k \zeta^k dm(\xi) dm(\zeta) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} (P z^n)(\xi) \bar{\xi}^k \zeta^{k-n} dm(\xi) dm(\zeta) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} (P z^n)(\xi) \bar{\xi}^k dm(\xi) \zeta^{k-n} dm(\zeta) \\ &= \int_{\mathbb{T}} (P z^n)(\xi) \bar{\xi}^k dm(\xi) \int_{\mathbb{T}} \zeta^{k-n} dm(\zeta) \\ &= \Lambda^k(P z^n) \delta_{nk}\end{aligned}$$

When $n \ge 0$, $z^n \in H^\infty$ and $P z^n = z^n$ because P is equal to identity on H^∞. Hence, $\Lambda^k(P^* z^n) = \Lambda^k(z^n)\delta_{nk} = (\delta_{nk})^2$ and $P^* z^n = z^n = P_+ z^n$. When $n < 0$, $P z^n \in H^\infty$. Thus, $\Lambda^n(P z^n) = 0$ and

$P^* z^n = 0 = P_+ z^n$.

□

Remark. As Λ^k is also a bounded functional in $(L^1)^*$, one can perform the same calculation in the proof of Theorem 2.15 to prove Theorem 2.8. Hence, the result about the Fourier multiplier operators is actually not necessary.

Theorem 2.16. *There does not exists any bounded projection from $C(\mathbb{T})$ onto $H^\infty \cap C(\mathbb{T})$.*

Proof. The proof is almost exactly the same as the one of Theorem 2.15. One just needs to replace L^∞ with $C(\mathbb{T})$.

□

2.3 First Application: Convergence of the Partial Fourier Sum in L^p for $1 < p < \infty$

In this section, we will apply the L^p boundedness of P_+ for $1 < p < \infty$ to show the convergence of the partial Fourier sum in L^p. We first need some definitions and lemmas from (2).

Definition 2.6 (Basis)**.** Let X be a Banach space and let $(x_k)_{k\in\mathbb{Z}} \subset X$ be a family indexed by the set of integers. It is called a symmetric basis if for all $x \in X$, there exists a unique sequence $(a_k(x))_{k\in\mathbb{Z}} \subset \mathbb{C}$ such that

$$x = \lim_{n\to\infty} \sum_{k=-n}^{n} a_k(x) x_k.$$

Remark. For $1 < p < \infty$, $(z^n)_{n\in\mathbb{Z}}$ forming a symmetric basis in L^p is equivalent to the convergence of the partial Fourier sum of f to f itself in L^p. Hence, our goal now is to show that $(z^n)_{n\in\mathbb{Z}}$ forms a symmetric basis, which explains why the following lemma is useful.

Definition 2.7 (Biorthogonal System)**.** A biorthogonal system is a pair $((x_n)_n \subset X, (f_k)_k \subset X^*)$ such that

$$f_k(x_n) = \delta_{kn}.$$

Lemma 2.17. *Given a biorthogonal system $((x_n)_n \subset X, (f_n)_n \subset X^*)$ in a Banach space X, set*

$$P_{n,m} = \sum_{k=m}^{n} f_k(\cdot) x_k, m \le n,$$
$$P_n = P_{n,-n}, n \ge 1.$$

Then $(x_n)_n$ is a symmetric basis iff $S := \sup_{n\ge1} \|P_n\| < \infty$ and span(x_n) is dense in X.

Proof. $\Rightarrow$ If $(x_n)_n$ is a symmetric basis, then for each x, there exists a unique sequence $(a_k(x))_{k\in\mathbb{Z}} \subset \mathbb{C}$ such that

$$x = \lim_{m\to\infty} \sum_{k=-m}^{m} a_k(x) x_k,$$
$$f_n(x) = \lim_{m\to\infty} \sum_{k=-m}^{m} a_k(x) f_n(x_k) = \lim_{m\to\infty} \sum_{k=-m}^{m} a_k(x)\delta_{kn} = a_n(x).$$

Hence, the sequence $P_n x$ converges to x. Since a convergent sequence is bounded, we have $\sup_{n\ge1} \|P_n x\| < \infty$. By the uniform boundedness principle, $\sup_{n\ge1} \|P_n\| < \infty$. Density of span$(x_n)$ is trivial.

$\Leftarrow$ We'll first show that $P_n x$ converges to x for each x. Notice that given any finite linear combination $z = \sum_{k\in K} c_k x_k$ of $(x_n)_n$, where K is a finite set. Then for sufficiently large n, we have

$$P_n(z) = P_n(\sum_{k\in K} c_k x_k) = \sum_{i=-n}^{n} f_i(\sum_{k\in K} c_k x_k) x_i = \sum_{k\in K}\sum_{i=-n}^{n} c_k \delta_{i,k} x_i = \sum_{k\in K} c_k x_k = z,$$

which implies that $P_n z$ converges to z for all $z \in \text{span}(x_n)$. Now for each $x \in X$, since $\text{span}(x_n)$ is dense in X, there exists a sequence $(z_k)_k \subset \text{span}(x_n)$ such that $z_k \to x$.

$$P_n(x) = P_n(\lim_{k\to\infty} z_k) = \lim_{k\to\infty} P_n(z_k).$$
$$\lim_{n\to\infty} P_n(z_k) = z_k.$$
$$\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty}\lim_{k\to\infty} P_n(z_k).$$

Notice that the first convergence is uniform in n because

$$\|P_n(x) - P_n(z_k)\| \leq \|P_n\|\|x - z_k\| \leq S\|x - z_k\|.$$

By the Moore-Osgood Theorem, we can interchange the limit operation to get

$$\lim_{n\to\infty} P_n(x) = \lim_{k\to\infty}\lim_{n\to\infty} P_n(z_k) = \lim_{k\to\infty} z_k = x.$$

We then need to show the uniqueness of the coordinate functionals. Suppose there exists another sequence $(a_k(x))_k \subset \mathbb{C}$ such that

$$\lim_{m\to\infty}\sum_{k=-m}^{m} f_k(x)x_k = \lim_{m\to\infty}\sum_{k=-m}^{m} a_k(x)x_k.$$

We apply f_n on both sides to get

$$f_n(\lim_{m\to\infty}\sum_{k=-m}^{m} f_k(x)x_k) = f_n(\lim_{m\to\infty}\sum_{k=-m}^{m} a_k(x)x_k),$$
$$\lim_{m\to\infty}\sum_{k=-m}^{m} f_k(x)f_n(x_k) = \lim_{m\to\infty}\sum_{k=-m}^{m} a_k(x)f_n(x_k),$$
$$\lim_{m\to\infty}\sum_{k=-m}^{m} f_k(x)\delta_{kn} = \lim_{m\to\infty}\sum_{k=-m}^{m} a_k(x)\delta_{kn},$$
$$f_n(x) = a_n(x).$$

□

Theorem 2.18. *For $1 < p < \infty$, $(z^n)_{n\in\mathbb{Z}}$ forms a symmetric basis in L^p. Consequently, for any $f \in L^p$, its Fourier series converges in L^p to f.*

Proof. $((z^n)_{n\in\mathbb{Z}}, (z^n)_{n\in\mathbb{Z}})$ is a biorthogonal system because

$$f \mapsto \int_{\mathbb{T}} f\bar{z}^n dm$$

is a continuous linear functional defined on L^p. By Lemma 2.17, we only need to show that $\sup_{n\geq 1}\|P_n\|_{L^p\to L^p} < \infty$ and $\text{span}(z^n)$ is dense in L^p. The second condition follows easily from the Stone Weierstrass theorem, so we only need to check the first condition. Suppose q is a trignometric polynomial. Then for each $n \geq 1$,

$$P_n q = z^{-n}P_{0,2n}(z^n q),$$
$$\|P_n q\|_{L^p} \leq \|P_{0,2n}(z^n q)\|_{L^p} \leq \|P_{0,2n}\|_{L^p\to L^p}\|z^n q\|_{L^p} = \|P_{0,2n}\|_{L^p\to L^p}\|q\|_{L^p}.$$

By density, we then have $\|P_n\|_{L^p\to L^p} \leq \|P_{0,2n}\|_{L^p\to L^p}$ and

$$\sup_{n\geq 1}\|P_n\|_{L^p\to L^p} \leq \sup_{n\geq 1}\|P_{0,2n}\|_{L^p\to L^p}.$$

Therefore, it suffices to show that $\sup_{n\geq 1} \|P_{0,2n}\|_{L^p\to L^p} < \infty$.

$$\begin{aligned}
P_{0,2n}(q) &= P_+q - \sum_{m=2n+1}^{\infty} (q, z^m)z^m \\
&= P_+q - z^{2n+1}\sum_{m=0}^{\infty}(q, z^{m+2n+1})z^m \\
&= P_+q - z^{2n+1}\sum_{m=0}^{\infty}(q\bar{z}^{2n+1}, z^m)z^m \\
&= P_+q - z^{2n+1}P_+(q\bar{z}^{2n+1}).
\end{aligned}$$

$$\|P_{0,2n}(q)\|_{L^p} \leq \|P_+q\|_{L^p} + \|P_+(q\bar{z}^{2n+1})\|_{L^p} \leq 2\|P_+\|_{L^p\to L^p}\|q\|_{L^p}.$$

Again by density and by Theorem 1.5, we then have $\|P_{0,2n}\|_{L^p\to L^p} \leq 2\|P_+\|_{L^p\to L^p} < \infty$.

□

Theorem 2.19. *The following definitions of H^p are equivalent for $1 < p < \infty$:*

1. $\{f \in L^p(\mathbb{T}) : \hat{f}(n) = 0, n < 0\}$.
2. $\{f \in L^p(\mathbb{T})$: *there exists* $(a_n)_{n\geq 0}$ *such that* $\sum_{n=0}^{k} a_n z^n$ *converges in* L^p *to* $f\}$.
3. *The closure of the span*$(z^n)_{n\geq 0}$ *in* $L^p(\mathbb{T})$.
4. *The image of* $L^p(\mathbb{T})$ *under* P_+.

Proof. $1 \Rightarrow 2$ Let $f \in L^p$ such that $\hat{f}(n) = 0$ for $n < 0$. We know that by Theorem 2.18, z^n is a (symmetric) basis for L^p, so f has the following series representation:

$$f = \sum_{n=0}^{\infty} \hat{f}(n)z^n.$$

This series converges in L^p norm.

$2 \Rightarrow 3$ Trivial.

$3 \Rightarrow 4$ Let f be in the closure of the span$(z^n)_{n\geq 0}$ such that $(f_n)_{n\geq 0} \subset \text{span}(z_n)_{n\geq 0}$ is a sequence that converges to $f \in L^p$. By Theorem 2.3, P_+ is bounded on L^p (or it is a bounded operator on $L^2 \cap L^p$ and thus can be extended to L^p). Since we clearly have $P_+f_n = f_n$, taking L^p limit on both sides gives:

$$P_+f = P_+(\lim f_n) = \lim P_+f_n = \lim f_n = f.$$

$4 \Rightarrow 1$ Suppose $g = P_+h$ for some $h \in L^p$. By Theorem 2.3, $g \in L^p$. We need to show that $\hat{g}(n) = 0$ for $n < 0$. Consider the series representation of g and h:

$$h = \sum_{n\in\mathbb{Z}} \hat{h}(n)z^n = \lim P_nh,$$

$$\sum_{n\in\mathbb{Z}} \hat{g}(n)z^n = g = P_+h = \lim P_+P_nh = \sum_{n\geq 0} \hat{h}(n)z^n.$$

Again $(z^n)_{n\in\mathbb{Z}}$ is a basis, so the uniqueness of the coefficients in the series representation implies that $\hat{g}(n) = 0$ for $n < 0$.

□

Remark: This theorem also shows that $(z^n)_{n\geq 0}$ is a basis for H^p, $1 < p < \infty$.

2.4 Second Application: Duality of Hardy Spaces

Theorem 2.20. *For every continuous linear functional $\phi \in (H^p)^*$, $1 < p < \infty$, there exists a unique $g \in H^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, such that*

$$\phi(f) = \int_{\mathbb{T}} f\bar{g}dm$$

for all $f \in H^p$. For each $g \in H^{p'}$, denote the integral operator $\int_{\mathbb{T}}(\cdot)\bar{g}dm$ by I_g, then the map $I : g \mapsto I_g$ gives an isomorphism between $H^{p'}$ and $(H^p)^*$.

Proof. By the Hahn-Banach theorem, $\phi \in (H^p)^*$ can be extended to a linear functional $\Phi \in (L^p)^*$. By the duality $(L^p)^* \cong L^{p'}$, there exists an $L^{p'}$ function g such that

$$\text{for all } f \in H^p, \phi(f) = \int_{\mathbb{T}} f\bar{g}dm.$$

Suppose first that $f \in \text{span}(z_n)_{n\geq 0} \subset H^2$ and $g \in L^2 \cap L^{p'}$. On L^2, P^+ is a projection operator. By the self-adjoint property of a projection operator defined on a Hilbert space,

$$\phi(f) = \int_{\mathbb{T}} f\bar{g}dm = \int_{\mathbb{T}} (P_+ f)\bar{g}dm,$$
$$\phi(f) = \int_{\mathbb{T}} f\overline{P_+ g}dm.$$

The equality then still holds for arbitrary $f \in H^2$ by the density of $\text{span}(z^n)_{n\geq 0}$ (proved in Theorem 1.10), and $P_+ g \in H^2$ indeed represents ϕ in this case. In general, if $g \in L^{p'}$, we taken a sequence $(g_n)_{n\geq 0} \subset L^2 \cap L^{p'}$ that converges to g in $L^{p'}$. Then for a fixed $f \in \text{span}(z^n)_{n\geq 0} \subset H^2 \cap H^p$,

$$\phi(f) = \int_{\mathbb{T}} (P_+ f)\bar{g}dm = \lim \int_{\mathbb{T}} (P_+ f)\overline{g_n}dm,$$
$$\phi(f) = \lim \int_{\mathbb{T}} f\overline{P_+ g_n}dm,$$
$$\phi(f) = \int_{\mathbb{T}} f\overline{P_+ g}dm,$$

where the last equality follows from the continuity of P_+ on $L^{p'}$. Lastly, for an arbitrary $f \in H^p$, we can find a sequence $(f_n)_{n\geq 0} \subset \text{span}(z^n)_{n\geq 0}$ that converges to f in L^p norm. Then,

$$\phi(f) = \lim \phi(f_n) = \lim \int_{\mathbb{T}} f_n \overline{P_+ g}dm = \int_{\mathbb{T}} f\overline{P_+ g}dm.$$

This convergence follows from the p'-integrability of $P_+ g$ and Hölder's inequality.

Now we prove the second part. We have already shown that I is surjective, and its boundedness follows easily from Hölder's inequality. It remains to show that I is injective (the open mapping theorem then guarantees that a bijective continuous operator is a homeomorphism). Suppose $I_g = I_h$ for two functions $g, h \in H^{p'}$. Then $I_g(z^n) = I_h(z^n)$ for any $n \geq 0$. By using the fact that $(z^n)_{n\in\mathbb{Z}}$ is a basis for $L^{p'}$, we have

$$I_g(z^n) = \int_{\mathbb{T}} z^n\bar{g}dm = \int_{\mathbb{T}} z^n\bar{h}dm = I_h(z^n),$$
$$\text{for all } n,\ \hat{g}(n) = \hat{h}(n),$$
$$g = \sum_{n\geq 0} \hat{g}(n)z^n = \sum_{n\geq 0} \hat{h}(n)z^n = h.$$

□

Definition 2.8. In order to describe the duality results of H^p for $p \notin (1, \infty)$, we introduce the following function spaces:

- $P_+L^\infty = \{f : \text{there exists } g \in L^\infty \text{ such that } f = P_+g\}$.
- $H^\infty_- = \{f \in L^\infty : \text{ for all } n \geq 0, \hat{f}(n) = 0\}$.
- $P_+C(\mathbb{T}) = \{f : \text{there exists } g \in C(\mathbb{T}) \text{ such that } f = P_+g\}$.
- $C(\mathbb{T})_- = \{f \in C(\mathbb{T}) : \text{ for all } n \geq 0, \hat{f}(n) = 0\}$.

If $g \in P_+L^\infty$ or $P_+C(\mathbb{T})$, then

$$\|g\|_{P_+L^\infty} = \inf\{\|h\|_{L^\infty} : P_+h = g\}.$$

The norms in H^∞_- and $C(\mathbb{T})_-$ are inherited from H^∞ and $C(\mathbb{T})$, respectively.

Theorem 2.21. *P_+L^∞ and $P_+C(\mathbb{T})$ are both Banach spaces. Hence, $P_+C(\mathbb{T})$ is a closed subspace of P_+L^∞.*

Proof. We first show that $\|\|_{P_+L^\infty}$ is indeed a norm. The only nontrivial part is to check that it is positive definite. Suppose

$$\|g\|_{P_+L^\infty} = 0$$

for some $g \in P_+L^\infty$. We can then find a sequence of functions $(h_n)_{n\geq 0} \subset L^\infty \subset L^2$ such that for each n,

$$P_+h_n = g, \|h_n\|_{L^\infty} \leq \frac{1}{n}.$$

The sequence $(h_n)_{n\geq 0}$ thus converges to 0 in L^2. Since P_+ is continuous from L^2 to L^2, we have

$$0 = P_+0 = P_+(\lim_{n\to\infty} h_n) = \lim_{n\to\infty} P_+h_n = \lim_{n\to\infty} g = g.$$

To check that P_+L^∞ is Banach, we just need to show that for every sequence $(x_k)_{k\geq 0} \subset P_+L^\infty$ such that $\sum_{k=1}^\infty \|x_k\|_{P_+L^\infty} < \infty$, $\sum_{k=1}^n x_k$ converges. For each k, we can find $y_k \in L^\infty$ such that

$$P_+y_k = x_k, \|y_k\|_{L^\infty} \leq \|x_k\|_{P_+L^\infty} + \frac{1}{2^k}.$$

The sequence $(y_k)_{k\geq 0}$ is absolutely convergent because

$$\sum_{k=1}^\infty \|y_k\|_{L^\infty} \leq \sum_{k=1}^\infty \|x_k\|_{P_+L^\infty} + \sum_{k=1}^\infty \frac{1}{2^k} = \sum_{k=1}^\infty \|x_k\|_{P_+L^\infty} + 1 < \infty.$$

Since L^∞ is a Banach space, $\sum_{k=1}^n y_k$ converges to some $y \in L^\infty$. Notice that $\sum_{k=1}^n x_k$ converges to P_+y because as $n \to \infty$, we have

$$\|P_+y - \sum_{k=1}^n x_k\|_{P_+L^\infty} = \|P_+(y - \sum_{k=1}^n y_k)\|_{P_+L^\infty} \leq \|y - \sum_{k=1}^n y_k\|_{L^\infty} \to 0.$$

The completeness of $P_+C(\mathbb{T})$ can be proved by using the same method.

□

Theorem 2.22. *For every continuous linear function $\phi \in (H^1)^*$, there exists a $g \in L^\infty$ such that for all $f \in H^1$,*

$$\phi(f) = \int_{\mathbb{T}} f\bar{g}dm.$$

Moreover, for all $p > 1$ and $f \in H^p$, we have

$$\phi(f) = \int_{\mathbb{T}} f\overline{P_+g}dm.$$

Remark. $p > 1$ is a necessary condition because $\overline{P_+ g}$ does not necessarily lie in L^∞ (recall in last subsection, we construct an explicit example of a bounded function g such that $P_+ g$ is unbounded), which prevents us from using Hölder's inequality to justify that the integral $\int_{\mathbb{T}} f \overline{P_+ g} dm$ is well-defined.

Proof. Again we can use the Hahn-Banach theorem to extend ϕ to obtain a bounded linear functional Φ defined on L^1. Since $(L^1)^* \cong L^\infty$, Φ is represented by some $g \in L^\infty$. Thus, for all $f \in H^1$,

$$\phi(f) = \Phi(f) = \int_{\mathbb{T}} f \bar{g} dm.$$

This functional proves the first part of this theorem. For the second part, we discuss two cases separately.

- When $p \geq 2$, an H^p function f also belongs to H^2. In addition, $g \in L^\infty \subset L^2$. Therefore, we can use the self-adjoint property of P_+ on L^2 to conclude that for all $f \in H^p$,

$$\phi(f) = \phi(P_+ f) = \int_{\mathbb{T}} P_+ f \bar{g} dm = \int_{\mathbb{T}} f \overline{P_+ g} dm.$$

- When $1 < p < 2$, let p' be the conjugate exponent of p. In this case, $P_+ g \in P_+ L^\infty \subset P_+ L^{p'} = H^{p'}$ and let $(f_n)_{n \geq 0} \subset H^p \cap H^2$ be a sequence of functions that converges in L^p to f. Then,

$$\phi(f) = \lim \phi(f_n) = \lim \int_{\mathbb{T}} f_n \overline{P_+ g} dm = \int_{\mathbb{T}} f \overline{P_+ g} dm,$$

  where the last equality follows from Hölder's inequality.

□

It turns out that $P_+ L^\infty$ is still isomorphic to $(H^1)^*$. We need some additional tools about quotient spaces before proving this fact.

Lemma 2.23 (First Isomorphism Theorem of Banach Space)**.** *Let X, Y be Banach spaces and $T : X \to Y$ a bounded surjective operator. Assume that $X/\ker(T)$ is equipped with canonical norm for quotient space. For each $x \in X$, denote its coset in $X/\ker(T)$ by $[x]$. Then the map $\widehat{T} : X/\ker(T) \to Y$ defined by*

$$\widehat{T}([x]) = Tx$$

is an isomorphism between Banach spaces.

Proof. See Exercise 35 in Chapter 5 of (4).

□

Corollary 2.23.1. *We have the following two isomorphisms:*

$$P_+ L^\infty \cong L^\infty / H^\infty_-,$$
$$P_+ C(\mathbb{T}) \cong C(\mathbb{T}) / C(\mathbb{T})_-.$$

Proof. By Lemma 2.23, it suffices to show that $H^\infty_- = \ker P_+ \cap L^\infty$ and $C(\mathbb{T})_- = \ker P_+ \cap C(\mathbb{T})$. For any $f \in L^\infty$ or $C(\mathbb{T})$, $f \in L^2$. Hence,

$$P_+ f = \sum_{n=0}^{\infty} \hat{f}(n) z^n,$$

where the series converges in L^2. Therefore, $P_+f = 0$ iff $\hat{f}(n) = 0$ for all $n \geq 0$ (the forward direction of this statement uses the uniqueness of the coefficients in the series representing 0).

□

Theorem 2.24. *For every functional $\phi \in (H^1)^*$, there exists a unique coset $[g] \in L^\infty/H^\infty_-$ such that for all $f \in H^1$,*

$$\phi(f) = \int_{\mathbb{T}} f\bar{g}dm.$$

For each $g \in L^\infty$, denote the the integral operator $\int_{\mathbb{T}}(\cdot)\bar{g}dm$ by I_g. $[g] \mapsto I_g$ is then an isomorphism from L^∞/H^∞_- to $(H^1)^*$. Thus, we have

$$P_+L^\infty \cong L^\infty/H^\infty_- \cong (H^1)^*.$$

Proof. To show that the map $[g] \mapsto I_g$ is well-defined, we need to show that for every element $h \in H^\infty_-$, I_h is the zero functional. For each $n \geq 0$,

$$I_h(z^n) = \overline{\hat{h}(n)} = 0$$

by definition. As $\text{span}(z^n)_{n\geq 0}$ is dense in H^1 by Theorem 1.5, I_h is indeed the zero functional.

$I_g \in (H^1)^*$ and $\|I_g\|_{H^1\to\mathbb{C}} \leq \|[g]\|_{L^\infty/H^\infty_-}$ by Hölder's inequality. In Theorem 2.22, we have already shown that the map $[g] \mapsto I_g$ is surjective. We also need to show that it is injective. Suppose g_1 and g_2 are two functions in L^∞ such that $I_{g_1} = I_{g_2}$. Then, for each $n \geq 0$,

$$\begin{aligned} I_{g_1}(z^n) &= I_{g_2}(z^n), \\ I_{g_1-g_2}(z^n) &= 0, \\ \overline{\widehat{g_1 - g_2}(n)} &= 0. \end{aligned}$$

This implies that $g_1 - g_2 \in H^\infty_-$. Hence, $[g_1] = [g_2]$.

□

A more explicit representation of a linear functional in $(H^1)^*$ is given in the following theorem.

Theorem 2.25. *For any $g \in P_+L^\infty$ and any $f \in H^1$, the following limit exists:*

$$I_g(f) := \lim_{r\nearrow 1} \int_{\mathbb{T}} f(r\zeta)\overline{g(r\zeta)}dm(\zeta).$$

I_g is a continuous operator defined on H^1 with its operator norm equivalent to $\|g\|_{P_+L^\infty}$. $g \to I_g$ is an isomorphism from P_+L^∞ to $(H^1)^*$.

Proof. This theorem is called Fefferman's duality theorem. The proof is rather long and complicated. Interested readers can refer to (9).

□

As an application, we show that P_+L^∞ equipped with the weak* topology is separable. We first state two lemmas.

Lemma 2.26. *Let X be a Banach space. Let $Y = X^*$ equipped with the weak* topology. Then $Y^* \cong X$.*

Proof. See Proposition 1.2 in Chapter 5 of (10).

□

Lemma 2.27. *Let Y be a locally convex space. A subspace E of Y is dense in Y iff for any continuous linear functional $f \in Y^*$, $f|_E \equiv 0$ implies that $f \equiv 0$.*

Proof. See Theorem 3.5 in Chapter 3 of (8).

□

Theorem 2.28. *The subspace* $\operatorname{span}(z^n)_{n\geq 0}$ *is weak* dense in P_+L^∞. Consequently, P_+L^∞ is separable when it is equipped with the weak* topology.*

Proof. By Theorem 2.25 and Lemma 2.26, we know that $(P_+L^\infty)^* \cong H^1$. Since the weak* topology is always locally convex, we can apply Lemma 2.27. Thus to prove the first part, it suffices to show that if $f \in H^1$ satisfies $I_g(f) = 0$ for all $g \in \operatorname{span}(z^n)_{n\geq 0}$, then $f = 0$. Notice that for all $n \geq 0$,

$$0 = I_{z^n}(f) = \lim_{r\nearrow 1} \int_{\mathbb{T}} f(r\zeta)\overline{(r\zeta)^n} dm(\zeta) = \int_{\mathbb{T}} f\overline{z^n} dm = \hat{f}(n)$$

because as $r \nearrow 1$, $f(r(\cdot))$ converges to f in L^1 and $r\zeta^n$ converges to ζ^n in L^∞. Since $f \in H^1$, its negative Fourier coefficients also vanish. As a result, we have

$$f * \Phi_n = 0 \xrightarrow{L^1} f.$$

We conclude that $f = 0$. Lastly, we show that the following countable set can be used to approximate elements of $\operatorname{span}(z^n)_{n\geq 0}$ in the weak* topology:

$$E := \{\sum_{k=0}^{n} (a_k + ib_k)z^k : n \in \mathbb{N}, a_k \in \mathbb{Q}, b_k \in \mathbb{Q}\}.$$

Indeed, for every nonempty weak* basic open set U, there are finite collections $(f_j)_{j=1}^k \subset L^1$, $(h_j)_{j=1}^k \subset P_+L^\infty$, and $(\varepsilon_j)_{j=1}^k$ such that

$$U = \bigcap_{j=1}^{k} \{g \in P_+L^\infty : |I_{g-h_j}(f_j)| < \varepsilon_j\}.$$

We need to show that U contains an element of E. We have already shown that $\operatorname{span}(z^n)_{n\geq 0}$ is weak* dense. Hence, U contains some polynomial $p(z) = \sum_{i=0}^t c_i z^i$ such that for each j,

$$|I_{p-h_j}(f_j)| < \frac{\varepsilon_j}{2}.$$

For each coefficient c_i of p, choose rationals a_i, b_i such that $d_i := a_i + ib_i$ and $|d_i - c_i| < \frac{\min_j\{\varepsilon_j\}}{2t \max_j\{\|f_j\|_{L^1}\}}$ (the case that $\max_j\{\|f_j\|_{L^1}\} = 0$ is trivial), and define another polynomial $q(z) := \sum_{i=0}^t d_i z^i \in E$. Then for each j, we have

$$\begin{aligned}
|I_{q-h_j}(f_j)| &\leq |I_{q-p}(f_j)| + |I_{p-h_j}(f_j)| \\
&< |\lim_{r\nearrow 1} \int_{\mathbb{T}} f_j(r\zeta)\overline{(q(r\zeta) - p(r\zeta))} dm(\zeta)| + \frac{\varepsilon_j}{2} \\
&= \int_{\mathbb{T}} |f_j||q - p| dm + \frac{\varepsilon_j}{2} \text{ (as } q - p \in L^\infty) \\
&\leq \int_{\mathbb{T}} |f_j| \sum_{i=0}^{t} |d_i - c_i| dm + \frac{\varepsilon_j}{2} \\
&< \int_{\mathbb{T}} |f_j| \frac{\min_j\{\varepsilon_j\}}{2 \max_j\{\|f_j\|_{L^1}\}} dm + \frac{\varepsilon_j}{2} \\
&< \frac{\varepsilon_j}{2} + \frac{\varepsilon_j}{2} \\
&< \varepsilon_j.
\end{aligned}$$

We conclude that $q \in U$.

$\square$

Theorem 2.29 (Riesz Brothers' Theorem)**.** *Let μ be a complex valued Borel measure on $\mathbb{T}$ such that for all $n \geq 1$,*

$$\int_{\mathbb{T}} z^n d\mu = 0$$

Then μ is absolutely continuous with respect to m and $d\mu = hdm$ for some $h \in H^1$.

Proof. See for instance (3) and (2).

$\square$

Theorem 2.30. *For every functional $\phi \in (C(\mathbb{T})/C(\mathbb{T})_-)^*$, there exists a unique function $g \in H^1$ such that for all $[f] \in C(\mathbb{T})/C(\mathbb{T})_-$,*

$$\phi([f]) = \int_{\mathbb{T}} f\bar{g}dm.$$

For each $g \in H^1$, denote the the integral operator $\int_{\mathbb{T}}(\cdot)\bar{g}dm$ by I_g. $g \mapsto I_g$ is then an isomorphism from H^1 to $(C(\mathbb{T})/C(\mathbb{T})_-)^*$. Consequently, we have

$$(P_+C(\mathbb{T}))^* \cong (C(\mathbb{T})/C(\mathbb{T})_-)^* \cong H^1.$$

Proof. We first show that I_g is well-defined on $C(\mathbb{T})/C(\mathbb{T})_-$. That is, we need to show that $I_g(h) = 0$ for any $h \in C(\mathbb{T})_-$. Since $C(\mathbb{T})_-$ is the closure of $\text{span}(z^n)_{n<0}$ with respect to the supremum norm, it suffices to show that $I_g(z^n) = 0$ for $n < 0$. As we assume that $g \in H^1$,

$$I_g(z^n) = \overline{\hat{g}(n)} = 0.$$

Again $I_g \in (C(\mathbb{T})/C(\mathbb{T})_-)^*$ and $\|I_g\|_{C(\mathbb{T})/C(\mathbb{T})_- \to \mathbb{C}} \leq \|g\|_{L^1}$ by Hölder's inequality. Now for each $\phi \in (C(\mathbb{T})/C(\mathbb{T})_-)^*$, define a linear functional $\psi \in (C(\mathbb{T}))^*$ by

$$\psi(f) = \phi([f]).$$

By the Riesz Representation theorem, there exists a complex measure μ such that for all $f \in C(\mathbb{T})$,

$$\psi(f) = \int_{\mathbb{T}} fd\mu.$$

Let $\bar{\mu}$ be the conjugate of μ. For all $n < 0$, $[z^n] = [0]$ in $C(\mathbb{T})/C(\mathbb{T})_-$. Let's calculate $\psi(z^n)$.

$$\psi(z^n) = \phi([z^n]) = \phi([0]) = 0.$$
$$\psi(z^n) = \int_{\mathbb{T}} z^n d\mu = \overline{\int_{\mathbb{T}} \bar{z}^n d\bar{\mu}}.$$

We conclude that the positive Fourier coefficients of $\bar{\mu}$ vanish. By Theorem 2.29, $d\bar{\mu} = gdm$ for some $g \in H^1$. We conclude that

$$\phi([f]) = \int_{\mathbb{T}} fd\mu = \overline{\int_{\mathbb{T}} \bar{f}d\bar{\mu}} = \overline{\int_{\mathbb{T}} \bar{f}gdm} = \int_{\mathbb{T}} f\bar{g}dm = I_g(f).$$

The map $g \mapsto I_g$ is thus surjective. Lastly, we claim that it is also injective. If $I_{g_1} = I_{g_2}$ for two functions $g_1, g_2 \in H^1$, then for each $n \in \mathbb{Z}$,

$$I_{g_1}(z^n) = I_{g_2}(z^n),$$
$$\overline{\hat{g_1}(n)} = \overline{\hat{g_2}(n)}.$$

Since L^1 functions can be approximated by the arithmetic means of the partial Fourier sums (see Theorem 1.5), two L^1 functions are equal a.e. iff they have the same Fourier coefficients. We conclude that $g_1 = g_2$.

$\square$

3 Hardy Space on the Hartogs Triangle

3.1 Basic Definitions and Motivations

Definition 3.1. The Hartogs triangle is defined to be the set

$$\mathbb{H} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$

Its distinguished boundary is the set

$$d_b(\mathbb{H}) := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\},$$

which can be identified with the torus $\mathbb{T}^2$.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain (i.e. open and connected) with a smooth boundary. In this case, it is easy to define the Hardy space $H^p(\Omega)$ and generalize some results in the case of $\Omega = \mathbb{D}$. This is because $\partial\Omega$ is still a smooth manifold after a small perturbation. More specifically, suppose $\rho : \mathbb{C}^n \to \mathbb{R}$ is a smooth defining function such that

$$\begin{aligned}
&\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}, \\
&\partial\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}, \\
&\nabla\rho \neq 0 \text{ on } \partial\Omega.
\end{aligned}$$

Then according to (11), we can define the Hardy space $H^p(\Omega)$ to be the set

$$\{f \in \mathrm{Hol}(\Omega) : \|f\|_{H^p(\Omega)} := \sup_{\varepsilon>0} \int_{\{\rho=-\varepsilon\}} |f|^p d\sigma_\varepsilon < \infty\},$$

where $d\sigma_\varepsilon$ is the Hausdorff measure defined on the boundary of $\Omega_\varepsilon := \{\rho = -\varepsilon\}$. This definition is natural given the definition of $H^p(\mathbb{D})$.

What if Ω does not have a smooth boundary? What is the natural definition of the Hardy space in this case? It is mentioned in (12) and (13) that the Hartogs triangle has a non-smooth boundary and many pathological behaviors, making it a well-known counterexample in several variable complex analysis. In (1), motivated by obtaining a reproducing kernel Hilbert space with the desired reproducing kernel, Monguzzi defines the Hardy space $H^2(\mathbb{H})$ of the Hartogs triangle to be

$$\left\{ f \in \mathrm{Hol}(\mathbb{H}) : \|f\|_{H^2(\mathbb{H})} := \sup_{(s,t)\in(0,1)\times(0,1)} \frac{1}{4\pi^2} \int_{d_b(\mathbb{H}_{st})} |f|^2 d\sigma_{st} < +\infty \right\},$$

where

$$\mathbb{H}_{st} = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|/s < |z_2| < t\right\}$$

and $d\sigma_{st}$ is the Hausdorff measured defined on the set $d_b(\mathbb{H}_{st}) := \{(z_1, z_2) : |z_1| = s, |z_2| = t\}$. Let $\Gamma := \{(j,k) \in \mathbb{Z}^2 : j \geq 0, k \geq -j-1\}$, which is called the set of allowable indices. Monguzzi shows that $(z_1^j z_2^k)_{(j,k)\in\Gamma}$ is an orthonormal basis of $H^2(\mathbb{H})$. That is, for any $f \in H^2(\mathbb{H})$, there are coefficients $(a_{jk})_{(j,k)\in\Gamma}$ such that

$$\|a_{jk}\|_{l^2} < \infty, f(z_1, z_2) = \sum_{(j,k)\in\Gamma} a_{jk} z_1^j z_2^k.$$

We can then associate f with a boundary value function $bf \in L^2(d_b(\mathbb{H}))$ defined as

$$bf(e^{i\theta}, e^{i\gamma}) = \sum_{(j,k)\in\Gamma} a_{jk} e^{ij\theta} e^{ik\gamma},$$

which converges absolutely exactly because $\|a_{jk}\|_{l^2} < \infty$. The Hardy space $H^2(d_b(\mathbb{H}))$ is then defined as

$$\{\sum_{(j,k)\in\Gamma}^{\infty} a_{jk} z_1^j z_2^k : \|a_{jk}\|_{l^2} < \infty\},$$

where the infinite sum converges in the L^2 norm. Notice that in this case, the order of summation is not important because we require the series to converge absolutely.

Monguzzi did not explicitly define $H^p(d_b(\mathbb{H}))$ for $1 \leq p \leq \infty$ in (1). Here's a natural definition of $H^p(d_b(\mathbb{H}))$ given the definition of $H^2(d_b(\mathbb{H}))$ and $H^p(\mathbb{T})$.

Definition 3.2. For $1 \leq p \leq \infty$,

$$H^p(d_b(\mathbb{H})) := \{f \in L^p(d_b(\mathbb{H})) : \hat{f}(j,k) = 0, (j,k) \notin \Gamma\}.$$

It is easy to see that they are Banach spaces by using the continuity of the Fourier coefficients.

Theorem 3.1. $\operatorname{span}(z_1^j z_2^k)_{(j,k)\in\Gamma}$ *is dense in* $H^p(d_b(\mathbb{H}))$ *for* $1 \leq p < \infty$ *and* $H^\infty(d_b(\mathbb{H})) \cap C(d_b(\mathbb{H}))$.

Proof. This is the higher dimensional analogue of Theorem 1.5. Again one uses the Fejér kernel to prove this theorem.

□

Theorem 3.2. *For* $1 < p < \infty$, $(z_1^j z_2^k)_{(j,k)\in\mathbb{Z}^2}$ *is a (symmetric) basis in* $L^p(d_b(\mathbb{H})) \cong L^p(\mathbb{T}^2)$, *in the sense that for every* $f \in L^p(d_b(\mathbb{H}))$, *there exists a unique sequence of coefficients* $(a_{jk})_{(j,k)\in\mathbb{Z}^2} = (\hat{f}(j,k))_{(j,k)\in\mathbb{Z}^2}$ *such that*

$$f = \lim_{n\to\infty} \sum_{\max(|j|,|k|)\leq n} \hat{f}(j,k) z_1^j z_2^k,$$

where the convergence is in L^p *for* $1 < p < \infty$.

Proof. See chapter 4 in (7) for the proof of convergence of this series (Grafakos actually uses induction to prove it for $\mathbb{T}^n$, where n can be any positive integer).

The uniqueness of coefficients follows from the continuity of the map $f \mapsto \hat{f}(j,k)$ with respect to the L^p norm.

□

Remark. By Theorem 3.2, it is easy to prove for $1 < p < \infty$, a function f in $L^p(d_b(\mathbb{H}))$ belongs to $H^p(d_b(\mathbb{H}))$ iff

$$f = \lim_{n\to\infty} \sum_{\substack{(j,k)\in\Gamma, \\ \max(|j|,|k|)\leq n}} \hat{f}(j,k) z_1^j z_2^k,$$

where the series converges in the L^p norm. This gives another characterization of $H^p(d_b(\mathbb{H}))$.

Denote the projection map from $L^2(d_b(\mathbb{H}))$ to $H^2(d_b(\mathbb{H}))$ by S , which is called the Szegő projection. In next section, we will prove that S can be extended to a bounded operator from $L^p(d_b(\mathbb{H}))$ to itself for $1 < p < \infty$.

3.2 Boundedness of the Szegő Projection

Let m_1, m_2 be the normalized Lebesgue measure on $\mathbb{T}$. Their product $m_1 \times m_2$ is the normalized Lebesgue measure on $\mathbb{T}^2$.

Theorem 3.3 (Monguzzi)**.** *For $1 < p < \infty$, S densely defined on $L^2(d_b(\mathbb{H})) \cap L^p(d_b(\mathbb{H}))$ can be extended to a bounded operator $S : L^p(d_b(\mathbb{H})) \to L^p(d_b(\mathbb{H}))$.*

Proof. We will follow Monguzzi's proof and elaborate the details. Define two functions $m_1, m_2 : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\begin{aligned} m_1(\xi, \eta) &= \frac{1 + \operatorname{sgn}(\xi)}{2}, \\ m_2(\xi, \eta) &= \frac{1 + \operatorname{sgn}(\xi + \eta + 1)}{2}. \end{aligned}$$

We can then decompose S into several operators:

$$\begin{aligned} T_1 f(z_1, z_2) &= \sum_{(j,k) \in \mathbb{Z}^2} m_1(j,k) \hat{f}(j,k) z_1^j z_2^k, \\ T_2 f(z_1, z_2) &= \sum_{(j,k) \in \mathbb{Z}^2} m_2(j,k) \hat{f}(j,k) z_1^j z_2^k, \\ T_3 f(z_1, z_2) &= \frac{1}{2} \sum_{k \geq 0} \hat{f}(0,k) z_2^k, \\ T_4 f(z_1, z_2) &= \frac{1}{2} \sum_{j \geq 1, j+k+1=0} \hat{f}(j,k) z_1^j z_2^k, \\ T_5 f(z_1, z_2) &= \frac{3}{4} \hat{f}(0,-1) z_2^{-1}. \end{aligned}$$

For any trignometric polynomial $f(z_1, z_2)$ defined on the torus $\mathbb{T}^2$, $T_i f$ is well-defined and

$$Sf = T_2(T_1 f) + T_3 f + T_4 f + T_5 f.$$

Therefore, it suffices to show that $\forall 1 \leq i \leq 5$, T_i is a bounded operator on L^p. T_5 is bounded because

$$\begin{aligned} \|T_5 f\|_{L^p} &= \left(\int_{\mathbb{T}^2} |\frac{3}{4} \hat{f}(0,-1)|^p dm_1 dm_2\right)^{\frac{1}{p}} \\ &= \frac{3}{4} |\hat{f}(0,-1)| \\ &= \frac{3}{4} \left| \int_{\mathbb{T}^2} f(z_1, z_2) \overline{z_2^{-1}} dm_1 dm_2 \right| \\ &\leq \frac{3}{4} \|f\|_{L^p}, \end{aligned}$$

where the last inequality follows from Hölder's inequality. I will show boundedness of other operators in the following lemmas.

□

Definition 3.3. Let $t_0 \in \mathbb{R}^2$. A bounded function m on $\mathbb{R}^2$ is called regulated at the point t_0 if

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{|t| \leq \varepsilon} (m(t_0 - t) - m(t_0)) dt = 0.$$

Definition 3.4. Let $1 < p < \infty$ and m be a bounded function on $\mathbb{R}^2$. It is called an L^p multiplier on $\mathbb{R}^2$ if the operator

$$T_m f := \mathcal{F}^{-1}(m\mathcal{F}f)$$

is bounded from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$. If m is a bounded function on $\mathbb{Z}^2$, then it is an L^p multiplier on $\mathbb{T}^2$ if the operator

$$T_m f(z_1, z_2) := \sum_{(j,k)\in\mathbb{Z}^2} m(j,k)\hat{f}(j,k)z_1^j z_2^k$$

is bounded from $L^p(\mathbb{T}^2)$ to $L^p(\mathbb{T}^2)$.

Remark. By using this definition, the boundedness of T_1 and T_2 from $L^p(\mathbb{T}^2)$ to $L^p(\mathbb{T}^2)$ is equivalent with the statement that T_1 and T_2 are L^p multipliers.

Lemma 3.4 (Transference of Multipliers)**.** *Suppose that $m : \mathbb{R}^2 \to \mathbb{C}$ is a regulated function at every point $(j,k) \in \mathbb{Z}^2$ and that m is an L^p multiplier on $\mathbb{R}^2$ for some $1 < p < \infty$. Then the sequence $(m(j,k))_{(j,k)\in\mathbb{Z}^2}$ is an L^p multiplier on the torus.*

Proof. See (7).

□

Lemma 3.5 (Halfplane Multiplier)**.** *Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $\nu \in \mathbb{R}^2$ a nonzero vector. Define $m_{x,\nu} : \mathbb{R}^2 \to \mathbb{R}$ by*

$$m_{x,\nu}(\xi,\eta) = sgn((\xi - x_1, \eta - x_2)\cdot\nu).$$

Then $(m_{x,\nu}(j,k))_{(j,k)\in\mathbb{Z}^2}$ is an L^p multiplier for $1 < p < \infty$.

Proof. $m_{x,\nu}$ is regulated at every point of the set $\mathbb{R}^2 - \{(\xi,\eta) : (\xi - x_1, \eta - x_2)\cdot\nu = 0\}$ because it is locally constant on this set. Let (ξ_0, η_0) be a point such that $(\xi_0 - x_1, \eta_0 - x_2)\cdot\nu = 0$ and let B be a square around it. The portion of B equal to -1 is the same as the portion of B equal to 1. Thus, the integral over B is zero and $m_{x,\nu}$ is regulated at (ξ_0, η_0). We conclude that $m_{x,\nu}$ is regulated everywhere.

By Lemma 3.4, it suffices to show that $m_{x,\nu}$ is an L^p multiplier for $\mathbb{R}^2$. Let $m := m_{(0,0),(1,0)}$. Notice that for any x, ν, there exists a rigid motion $R : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $R(\xi,\eta) = A((\xi,\eta) - b) = A(\xi,\eta) - Ab$, where A is an orthogonal matrix and $b \in \mathbb{R}^2$ is a vector, such that $m_{x,\nu} = m \circ R$. We first show that the L^p boundedness of $m_{x,\nu}$ follows from the L^p boundedness of $m_{(0,0),(1,0)}$. Let f be a compactly supported C^∞ function on $\mathbb{R}^2$. We will use the following properties of (inverse) Fourier transform:

$$\begin{aligned}
\mathcal{F}(\tau^b f)(\xi,\eta) &= e^{-2\pi i b\cdot(\xi,\eta)}\mathcal{F}(f)(\xi,\eta),\\
\mathcal{F}(f\circ A)(\xi,\eta) &= \mathcal{F}(f)(A(\xi,\eta)),\\
\mathcal{F}^{-1}(\tau^b f)(\xi,\eta) &= e^{2\pi i b\cdot(\xi,\eta)}\mathcal{F}^{-1}(f)(\xi,\eta),\\
\mathcal{F}^{-1}(f\circ A)(\xi,\eta) &= \mathcal{F}^{-1}(f)(A(\xi,\eta)).
\end{aligned}$$

Also recall that the Lebesgue measure (and thus the value of integral) is invariant under rigid motions. If we assume that m is an L^p multiplier, then we have

$$\begin{aligned}
\|\mathcal{F}^{-1}(m_{x,\nu}\mathcal{F}f)\|_{L^p} &= \|\mathcal{F}^{-1}((m\circ R)\mathcal{F}f)\|_{L^p}\\
&= \|\mathcal{F}^{-1}((m\circ R)\mathcal{F}(f\circ R^{-1}\circ R))\|_{L^p}\\
&= \|\mathcal{F}^{-1}((m\cdot\mathcal{F}(f\circ R^{-1}))\circ R)\|_{L^p}\ \text{(define } g := m\cdot\mathcal{F}(f\circ R^{-1})\text{)}\\
&= \|\mathcal{F}^{-1}(g\circ R)\|_{L^p}\\
&= \|\mathcal{F}^{-1}(\tau^b(g\circ A))\|_{L^p}\\
&= \|e^{2\pi i b\cdot(\xi,\eta)}\mathcal{F}^{-1}(g\circ A)\|_{L^p}\\
&= \|\mathcal{F}^{-1}(g)\circ A\|_{L^p}\\
&= \|\mathcal{F}^{-1}(g)\|_{L^p}\\
&= \|\mathcal{F}^{-1}(m\cdot\mathcal{F}(f\circ R^{-1}))\|_{L^p}\\
&\lesssim \|f\circ R^{-1}\|_{L^p}\ \text{(since we assume that } m \text{ is an } L^p \text{ multiplier)}\\
&= \|f\|_{L^p}.
\end{aligned}$$

This calculation shows that $m_{x,\nu}$ is an L^p multiplier. Therefore, our goal now is to prove that $m(\xi,\eta) = \operatorname{sgn}(\xi)$ is an L^p multiplier. Denote the Hilbert transform on $\mathbb{R}$ by H. It is known that H is a bounded operator from $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ (see (7)). For all $z_1, z_2 \in \mathbb{R}$, define $f_{z_2}(z_1) = f(z_1, z_2)$. Denote the one dimensional Fourier transform and inverse Fourier transform on $\mathbb{R}$ by $\widehat{\ }$ and $\vee$. We can then do a formal calculation by using Fubini's theorem and the Fourier inversion theorem:

$$\begin{aligned}
\mathcal{F}^{-1}(m\mathcal{F}f)(\xi,\eta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(y_1)\mathcal{F}f(y_1,y_2)e^{2\pi i(y_1\xi+y_2\eta)}dy_1dy_2\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(y_1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(z_1,z_2)e^{-2\pi i(z_1y_1+z_2y_2)}dz_1dz_2e^{2\pi i(y_1\xi+y_2\eta)}dy_1dy_2\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(y_1)f(z_1,z_2)e^{-2\pi iz_1y_1}e^{-2\pi iz_2y_2}e^{2\pi iy_1\xi}e^{2\pi iy_2\eta}dz_1dz_2dy_1dy_2\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(y_1)\int_{\mathbb{R}} f_{z_2}(z_1)e^{-2\pi iz_1y_1}dz_1e^{-2\pi iz_2y_2}e^{2\pi iy_1\xi}e^{2\pi iy_2\eta}dz_2dy_1dy_2\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(y_1)\widehat{f_{z_2}}(y_1)e^{-2\pi iz_2y_2}e^{2\pi iy_1\xi}e^{2\pi iy_2\eta}dz_2dy_1dy_2\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(y_1)\widehat{f_{z_2}}(y_1)e^{2\pi iy_1\xi}dy_1e^{-2\pi iz_2y_2}e^{2\pi iy_2\eta}dz_2dy_2\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} (\operatorname{sgn}(y_1)\widehat{f_{z_2}}(y_1))^{\vee}(\xi)e^{-2\pi iz_2y_2}e^{2\pi iy_2\eta}dz_2dy_2\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} (iHf_{z_2})(\xi)e^{-2\pi iz_2y_2}e^{2\pi iy_2\eta}dz_2dy_2\ \text{(define } s(z_2) := (iHf_{z_2})(\xi)\text{)}\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} s(z_2)e^{-2\pi iz_2y_2}dz_2e^{2\pi iy_2\eta}dy_2\\
&= \int_{\mathbb{R}} \hat{s}(y_2)e^{2\pi iy_2\eta}dy_2\\
&= (\hat{s})^{\vee}(\eta)\\
&= s(\eta)\\
&= (iHf_\eta)(\xi).
\end{aligned}$$

Notice that in the last part of our calculations, we assume that the Fourier inversion holds for s. We justify it by showing that both s and $\hat{s}$ are in the Schwartz class $\mathcal{S}(\mathbb{R})$ in the next several lemmas.

Lastly, we can show the L^p boundedness of m:

$$\begin{aligned}
\|\mathcal{F}^{-1}(m\mathcal{F}f)\|_{L^p(\mathbb{R}^2)} &= (\int_{\mathbb{R}}\int_{\mathbb{R}} |(iHf_\eta)(\xi)|^p d\xi d\eta)^{\frac{1}{p}} \\
&= (\int_{\mathbb{R}} \|Hf_\eta\|_{L^p}^p d\eta)^{\frac{1}{p}} \\
&\lesssim (\int_{\mathbb{R}} \|f_\eta\|_{L^p}^p d\eta)^{\frac{1}{p}} \\
&= (\int_{\mathbb{R}}\int_{\mathbb{R}} |f_\eta(\xi)|^p d\xi d\eta)^{\frac{1}{p}} \\
&= (\int_{\mathbb{R}}\int_{\mathbb{R}} |f(\xi,\eta)|^p d\xi d\eta)^{\frac{1}{p}} \\
&= \|f\|_{L^p(\mathbb{R}^2)}.
\end{aligned}$$

Since compactly supported functions are dense in $L^p(\mathbb{R}^2)$ for $1 < p < \infty$, the above inequality is true for any $f \in L^p(\mathbb{R}^2)$.

□

Before going through the lemmas, let's first introduce Grafakos' notation. Let u be any tempered distribution and $f \in \mathcal{S}(\mathbb{R})$.

$$\begin{aligned}
\text{For all } n, m \geq 0, \|f\|_{n,m} &:= \|x^n f^{(m)}(x)\|_{L^\infty} \\
\langle u, f\rangle &:= u(f), \\
\widetilde{f}(x) = f^{\sim}(x) &:= f(-x), \\
\text{For all } \xi \in \mathbb{R}, \tau^\xi f(x) &:= f(x-\xi).
\end{aligned}$$

Lemma 3.6. *Fix $\xi \in \mathbb{R}$. The maps $f \mapsto \tilde{f}$ and $f \mapsto \tau^\xi f$ are continuous endomorphisms on $\mathcal{S}(\mathbb{R})$.*

Proof. $\mathcal{S}(R)$ is a Fréchet space. Hence, continuity of a function follows from the control of each seminorm by a finite collection of seminorms. For all $n, m \geq 0$,

$$\begin{aligned}
\|\tilde{f}\|_{n,m} &= \|x^n(\tilde{f})^{(m)}(x)\|_{L^\infty} \\
&= \|(-x)^n f^{(m)}(-x)\|_{L^\infty} \\
&= \|x^n f^{(m)}(x)\|_{L^\infty} \\
&= \|f\|_{n,m}, \\
\|\tau^\xi f\|_{n,m} &= \|x^n(\tau^\xi f)^{(m)}(x)\|_{L^\infty} \\
&= \|x^n f^{(m)}(x-\xi)\|_{L^\infty} \\
&= \|(x+\xi)^n f^{(m)}(x)\|_{L^\infty} \\
&\leq \sum_{k=0}^{n} \binom{n}{k} |\xi|^{n-k} \|x^k f^{(m)}(x)\|_{L^\infty} \\
&= \sum_{k=0}^{n} \binom{n}{k} |\xi|^{n-k} \|f\|_{k,m}.
\end{aligned}$$

□

Lemma 3.7. *Let f be a smooth compactly supported function defined on $\mathbb{R}^2$. Then for each $z_2 \in \mathbb{R}$,*

$$\frac{f_{z_2+h} - f_{z_2}}{h} - (\partial_y f)_{z_2}$$

converges to 0 *in* $\mathcal{S}(\mathbb{R})$ *as* h *goes to* 0, *where* $\partial_y f$ *is the partial derivative of* f *with respect to the second variable.*

Proof. Let $\partial_x f$ be the partial derivative of f with respect to the first variable. For each $n, m \geq 0$ and $u \in \mathbb{R}$, by the Mean value theorem, there exists some $y_0 \in [z_2, z_2 + h]$ such that

$$\begin{aligned} u^n \Big[\frac{f_{z_2+h} - f_{z_2}}{h} - (\partial_y f)_{z_2} \Big]^{(m)}(u) &= u^n \Big[\frac{(\partial_x^m f)_{z_2+h} - (\partial_x^m f)_{z_2}}{h} - (\partial_x^m \partial_y f)_{z_2} \Big](u) \\ &= u^n \Big[\frac{(\partial_x^m f)(u, z_2 + h) - (\partial_x^m f)(u, z_2)}{h} - (\partial_x^m \partial_y f)(u, z_2) \Big] \\ &= u^n \Big[(\partial_x^m \partial_y f)(u, y_0) - (\partial_x^m \partial_y f)(u, z_2) \Big]. \end{aligned}$$

Since the function $(a, b) \mapsto a^n (\partial_x^m \partial_y f)(a, b)$ is uniformly continuous and the distance between y_0 and z_2 decreases as $h \to 0$, we have

$$\begin{aligned} \| \frac{f_{z_2+h} - f_{z_2}}{h} - (\partial_y f)_{z_2} \|_{n,m} &= \| u^n \Big[\frac{f_{z_2+h} - f_{z_2}}{h} - (\partial_y f)_{z_2} \Big]^{(m)}(u) \|_{L^\infty} \\ &= \| u^n \Big[(\partial_x^m \partial_y f)(u, y_0(u)) - (\partial_x^m \partial_y f)(u, z_2) \Big] \|_{L^\infty} \\ &\to 0. \end{aligned}$$

□

Lemma 3.8. *Let* f *be a smooth compactly supported function defined on* $\mathbb{R}^2$. *For each* $\xi \in \mathbb{R}$, $s(z_2) := (Hf_{z_2})(\xi)$ *is also smooth and compactly supported. Thus,* s *and* $\hat{s}$ *are Schwartz functions, and the Fourier inversion theorem holds for* s.

Proof. For z_2 large enough, $f_{z_2} \equiv 0$. Hence, $Hf_{z_2} \equiv 0$, which implies that s is compactly supported.

We now show that

$$s'(z_2) = (H(\partial_y f)_{z_2})(\xi).$$

If this is true, then the smoothness of s follows from induction (replace f by $\partial_y f$ in this formula). By Lemma 3.6 and Lemma 3.7, we have the following convergence in $\mathcal{S}(\mathbb{R})$:

$$\lim_{h \to 0} \tau^\xi (\frac{f_{z_2+h} - f_{z_2}}{h})^\sim = \tau^\xi (\widetilde{(\partial_y f)_{z_2}}).$$

We know that H is given by convolution with a tempered distribution W. We are now ready to do the following calculation:

$$\begin{aligned} \lim_{h \to 0} \frac{s(z_2 + h) - s(z_2)}{h} &= \lim_{h \to 0} \frac{(Hf_{z_2+h})(\xi) - (Hf_{z_2})(\xi)}{h} \\ &= \lim_{h \to 0} [H(\frac{f_{z_2+h} - f_{z_2}}{h})](\xi) \\ &= \lim_{h \to 0} \langle W, \tau^\xi (\widetilde{\frac{f_{z_2+h} - f_{z_2}}{h}}) \rangle \\ &= \langle W, \lim_{h \to 0} \tau^\xi (\widetilde{\frac{f_{z_2+h} - f_{z_2}}{h}}) \rangle \\ &= \langle W, \tau^\xi (\widetilde{(\partial_y f)_{z_2}}) \rangle \\ &= (H(\partial_y f)_{z_2})(\xi). \end{aligned}$$

□

Lemma 3.9. *T_1 and T_2 are L^p multipliers for $1 < p < \infty$.*

Proof. For $i = 1, 2$, T_i is a sum of two bounded operators.

$$T_1 f = \frac{1}{2} f + \frac{1}{2} T_{m_{(0,0),(1,0)}} f.$$
$$T_2 f = \frac{1}{2} f + \frac{1}{2} T_{m_{(-1,0),(1,1)}} f.$$

□

Lemma 3.10 (Line Multiplier)**.** *Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $\nu \in \mathbb{R}^2$ be a nonzero vector. Define $m : \mathbb{R}^2 \to \mathbb{R}$ by*

$$m^x_\nu(\xi, \eta) = \begin{cases} 1, & (\xi - x_1, \eta - x_2) \cdot \nu = 0 \\ 0, & \textit{Otherwise} \end{cases}.$$

Then $(m^x_\nu(j, k))_{(j,k)\in\mathbb{Z}^2}$ is an L^p multiplier for $1 < p < \infty$.

Proof. We make the following observation:

$$m^x_\nu(\xi, \eta) = 1 - m^2_{x,\nu}(\xi, \eta),$$
$$T_{m^x_\nu} f = f - T_{m_{x,\nu}}(T_{m_{x,\nu}} f),$$

where $T_{m_{x,\nu}}$ is an L^p multiplier by Lemma 3.5.

□

Lemma 3.11 (Halfline Multiplier)**.** *Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two points in $\mathbb{R}^2$. Let $\nu_1, \nu_2 \in \mathbb{R}^2$ be two nonzero vectors. Define $m : \mathbb{R}^2 \to \mathbb{R}$ by*

$$m^{x,y}_{\nu_1,\nu_2}(\xi, \eta) = \begin{cases} 1, & (\xi - x_1, \eta - x_2) \cdot \nu_1 = 0 \textit{ and } (\xi - y_1, \eta - y_2) \cdot \nu_2 \geq 0 \\ 0, & \textit{Otherwise} \end{cases}.$$

Then $(m^{x,y}_{\nu_1,\nu_2}(j, k))_{(j,k)\in\mathbb{Z}^2}$ is an L^p multiplier for $1 < p < \infty$.

Proof. Notice that

$$m^{x,y}_{\nu_1,\nu_2}(\xi, \eta) = \frac{(m_{y,\nu_2}(\xi, \eta) + 1)}{2} m^x_{\nu_1}(\xi, \eta) + \frac{1}{2} m^y_{\nu_2}(\xi, \eta) m^x_{\nu_1}(\xi, \eta),$$
$$T_{m^{x,y}_{\nu_1,\nu_2}} f = \frac{1}{2} T_{m_{y,\nu_2}}(T_{m^x_{\nu_1}} f) + \frac{1}{2} T_{m^x_{\nu_1}} f + \frac{1}{2} T_{m^y_{\nu_2}}(T_{m^x_{\nu_1}} f).$$

where $T_{m^x_{\nu_1}}, T_{m^y_{\nu_2}}$ are L^p multipliers by Lemma 3.10, and $T_{m_{y,\nu_2}}$ is an L^p multipliers by Lemma 3.5.

□

Lemma 3.12. *T_3 and T_4 are L^p multipliers for $1 < p < \infty$.*

Proof. We write these two operators as halfline multipliers:

$$T_3 f = \frac{1}{2} T_{m^{(0,0),(0,0)}_{(1,0),(0,1)}} f,$$
$$T_4 f = \frac{1}{2} T_{m^{(-1,0),(1,0)}_{(1,1),(1,0)}} f.$$

□

Theorem 3.13. *There does not exist any bounded projection from $L^1(d_b(\mathbb{H}))$ to $H^1(d_b(\mathbb{H}))$.*

Proof. See (6), (14), (15), and chapter 3 of (16). $\square$

Theorem 3.14. *The following definitions of $H^p(d_b(\mathbb{H}))$ are equivalent for $1 < p < \infty$.*

1. $\{f \in L^p(d_b(\mathbb{H})) : \hat{f}(j,k) = 0, (j,k) \notin \Gamma\}$.
2. $\{f \in L^p(d_b(\mathbb{H})) : f = \lim_{n\to\infty} \sum_{\substack{(j,k)\in\Gamma,\\ \max(|j|,|k|)\le n}} \hat{f}(j,k) z_1^j z_2^k,$ *which converges in* $L^p\}$.
3. *The closure of the span*$(z_1^j z_2^k)_{(j,k)\in\Gamma}$ *in* $L^p(d_b(\mathbb{H}))$.
4. *The image of* $L^p(d_b(\mathbb{H}))$ *under* S.

Remark. This is the analogue of Theorem 2.19.

Proof. $1 \Rightarrow 2$ See the remark in section 3.1.

$2 \Rightarrow 3$ Trivial.

$3 \Rightarrow 4$ Suppose f is in the closure of the $\text{span}(z_1^j z_2^k)_{(j,k)\in\Gamma}$ such that $(f_n)_{n\ge 0} \subset \text{span}(z_1^j z_2^k)_{(j,k)\in\Gamma}$ is a sequence that converges to f in L^p norm. By Theorem 3.3, S is bounded on $L^p(d_b(\mathbb{H}))$. Since we clearly have $Sf_n = f_n$, taking L^p limit on both sides gives:

$$Sf = S(\lim f_n) = \lim Sf_n = \lim f_n = f.$$

$4 \Rightarrow 1$ Suppose $g = Sh$ for some $h \in L^p(d_b(\mathbb{H}))$. By Theorem 3.3, $g \in L^p(d_b(\mathbb{H}))$. We need to show that $\hat{g}(j,k) = 0$ for $(j,k) \notin \Gamma$. Consider the series representation of g and h:

$$h = \lim_{n\to\infty} \sum_{\max(|j|,|k|)\le n} \hat{h}(j,k) z_1^j z_2^k,$$

$$\lim_{n\to\infty} \sum_{\max(|j|,|k|)\le n} \hat{g}(j,k) z_1^j z_2^k = g = Sh = \lim_{n\to\infty} \sum_{\substack{\max(|j|,|k|)\le n,\\ (j,k)\in\Gamma}} \hat{h}(j,k) z_1^j z_2^k$$

Again $(z_1^j z_2^k)_{(j,k)\in\mathbb{Z}^2}$ is a basis by Theorem 3.2, so the uniqueness of the coefficients in the series representation implies that $\hat{g}(j,k) = 0$ for $(j,k) \notin \Gamma$. $\square$

3.3 Duality of Hardy Spaces of the Hartogs Triangle

Theorem 3.15. *For every continuous linear functional $\phi \in (H^p(d_b(\mathbb{H})))^*$, $1 < p < \infty$, there exists a unique $g \in H^{p'}(d_b(\mathbb{H}))$, $\frac{1}{p} + \frac{1}{p'} = 1$, such that*

$$\phi(f) = \int_{\mathbb{T}^2} f\bar{g}d(m_1 \times m_2)$$

for all $f \in H^p(d_b(\mathbb{H}))$. For each $g \in H^{p'}$, denote the integral operator $\int_{\mathbb{T}^2}(\cdot)\bar{g}d(m_1 \times m_2)$ by I_g, then the map $I : g \mapsto I_g$ gives an isomorphism between $H^{p'}(d_b(\mathbb{H}))$ and $(H^p(d_b(\mathbb{H})))^*$.

Proof. By the Hahn-Banach theorem, $\phi \in (H^p(d_b(\mathbb{H})))^*$ can be extended to a linear functional $\Phi \in (L^p(d_b(\mathbb{H})))^*$. By the duality $(L^p(d_b(\mathbb{H})))^* \cong L^{p'}(d_b(\mathbb{H}))$, there exists an $L^{p'}(d_b(\mathbb{H}))$ function g such that

$$\text{for all } f \in H^p(d_b(\mathbb{H})), \phi(f) = \int_{\mathbb{T}^2} f\bar{g}d(m_1 \times m_2).$$

We taken a sequence $(g_n)_{n\geq 0} \subset L^2(d_b(\mathbb{H})) \cap L^{p'}(d_b(\mathbb{H}))$ that converges to g in $L^{p'}(d_b(\mathbb{H}))$. Then for a fixed $f \in \text{span}(z_1^j z_2^k)_{(j,k)\in\Gamma} \subset H^2(d_b(\mathbb{H})) \cap H^p(d_b(\mathbb{H}))$, we can use the self-adjoint property of S on $L^2(d_b(\mathbb{H}))$ to show that

$$\phi(f) = \int_{\mathbb{T}^2} (Sf)\bar{g}d(m_1 \times m_2) = \lim \int_{\mathbb{T}^2} (Sf)\overline{g_n}d(m_1 \times m_2),$$
$$\phi(f) = \lim \int_{\mathbb{T}^2} f\overline{Sg_n}d(m_1 \times m_2),$$
$$\phi(f) = \int_{\mathbb{T}^2} f\overline{Sg}d(m_1 \times m_2),$$

where the last equality follows from the continuity of S on $L^{p'}(d_b(\mathbb{H}))$. Lastly, for an arbitrary $f \in H^p(d_b(\mathbb{H}))$, we can find a sequence $(f_n)_{n\geq 0} \subset \text{span}(z_1^j z_2^k)_{(j,k)\in\Gamma}$ that converges to f in L^p norm. Then,

$$\phi(f) = \lim \phi(f_n) = \lim \int_{\mathbb{T}^2} f_n\overline{Sg}d(m_1 \times m_2) = \int_{\mathbb{T}^2} f\overline{Sg}d(m_1 \times m_2).$$

This convergence follows from the p'-integrability of Sg and Hölder's inequality.

Now we prove the second part. We have already shown that I is surjective, and the boundedness of I follows easily from Hölder's inequality. It remains to show that I is injective. Suppose $I_g = I_h$ for two functions $g, h \in H^{p'}(d_b(\mathbb{H}))$. Then $I_g(z_1^j z_2^k) = I_h(z_1^j z_2^k)$ for any $(j,k) \in \Gamma$. By using the fact that $(z_1^j z_2^k)_{(j,k)\in\mathbb{Z}^2}$ is a basis for $L^{p'}(d_b(\mathbb{H}))$, we have

$$I_g(z_1^j z_2^k) = \int_{\mathbb{T}^2} z_1^j z_2^k \bar{g}d(m_1 \times m_2) = \int_{\mathbb{T}^2} z_1^j z_2^k \bar{h}d(m_1 \times m_2) = I_h(z_1^j z_2^k),$$
$$\hat{g}(j,k) = \hat{h}(j,k),$$
$$g = \lim_{n\to\infty} \sum_{\substack{\max(|j|,|k|)\leq n,\\ (j,k)\in\Gamma}} \hat{g}(j,k) z_1^j z_2^k = \lim_{n\to\infty} \sum_{\substack{\max(|j|,|k|)\leq n,\\ (j,k)\in\Gamma}} \hat{h}(j,k) z_1^j z_2^k = h.$$

□

Definition 3.5. Define

- $SL^\infty(d_b(\mathbb{H})) = \{f : \text{there exists } g \in L^\infty(d_b(\mathbb{H})) \text{ such that } f = Sg\}$.
- $H^\infty_-(d_b(\mathbb{H})) = \{f \in L^\infty(d_b(\mathbb{H})) : \text{ for all } (j,k) \in \Gamma, \hat{f}(j,k) = 0\}$.

If $g \in SL^\infty(d_b(\mathbb{H}))$, then

$$\|g\|_{SL^\infty} = \inf\{\|h\|_{L^\infty} : Sh = g\}.$$

Lemma 3.16. *We have the following isomorphism:*

$$SL^\infty(d_b(\mathbb{H})) \cong L^\infty(d_b(\mathbb{H}))/H^\infty_-(d_b(\mathbb{H})).$$

Proof. By Lemma 2.23, it suffices to show that $H^\infty_-(d_b(\mathbb{H})) = \ker S \cap L^\infty(d_b(\mathbb{H}))$. For any $f \in L^\infty(d_b(\mathbb{H}))$, $f \in L^2(d_b(\mathbb{H}))$. Hence,

$$Sf = \lim_{n\to\infty} \sum_{\substack{\max(|j|,|k|)\leq n,\\ (j,k)\in\Gamma}} \hat{f}(j,k) z_1^j z_2^k,$$

where the series converges in L^2. Therefore, $Sf = 0$ iff $\hat{f}(j,k) = 0$ for all $(j,k) \in \Gamma$.

□

Theorem 3.17. *For every functional $\phi \in (H^1(d_b(\mathbb{H})))^*$, there exists a unique coset $[g] \in L^\infty(d_b(\mathbb{H}))/H^\infty_-(d_b(\mathbb{H}))$ such that for all $f \in H^1(d_b(\mathbb{H}))$,*

$$\phi(f) = \int_{\mathbb{T}^2} f\bar{g} d(m_1 \times m_2).$$

For each $g \in L^\infty(d_b(\mathbb{H}))$, denote the the integral operator $\int_{\mathbb{T}^2}(\cdot)\bar{g}d(m_1 \times m_2)$ by I_g. $[g] \mapsto I_g$ is then an isomorphism from $L^\infty(d_b(\mathbb{H}))/H^\infty_-(d_b(\mathbb{H}))$ to $(H^1(d_b(\mathbb{H})))^*$. Thus, we have

$$SL^\infty(d_b(\mathbb{H})) \cong L^\infty(d_b(\mathbb{H}))/H^\infty_-(d_b(\mathbb{H})) \cong (H^1(d_b(\mathbb{H})))^*.$$

Proof. To show that the map $[g] \mapsto I_g$ is well-defined, we need to show that for every element $h \in H^\infty_-(d_b(\mathbb{H}))$, I_h is the zero functional. For each $(j,k) \in \Gamma$,

$$I_h(z_1^j z_2^k) = \overline{\hat{h}(j,k)} = 0$$

by definition. As $\text{span}(z_1^j z_2^k)_{(j,k)\in\Gamma}$ is dense in H^1 by Theorem 3.1, I_h is indeed the zero functional.

$I_g \in (H^1(d_b(\mathbb{H})))^*$ and $\|I_g\|_{H^1\to\mathbb{C}} \leq \|[g]\|_{L^\infty(d_b(\mathbb{H}))/H^\infty_-(d_b(\mathbb{H}))}$ by Hölder's inequality. The map $[g] \mapsto I_g$ is surjective by the Hahn-Banach theorem. We also need to show that it is injective. Suppose g_1 and g_2 are two functions in $L^\infty(d_b(\mathbb{H}))$ such that $I_{g_1} = I_{g_2}$. Then, for each $(j,k) \in \Gamma$,

$$\begin{aligned} I_{g_1}(z_1^j z_2^k) &= I_{g_2}(z_1^j z_2^k), \\ I_{g_1-g_2}(z_1^j z_2^k) &= 0, \\ \overline{\widehat{g_1-g_2}(j,k)} &= 0. \end{aligned}$$

This implies that $g_1 - g_2 \in H^\infty_-$. Hence, $[g_1] = [g_2]$.

□

Remark. It would be desirable to also prove the analogue of Theorem 2.30, but it is not clear how one should generalize the Riesz Brothers' Theorem to higher dimensions, which can be a potential research topic for the future.

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