Conormal Chow Rings of Uniform Matroids

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1 Introduction

Starting with a matroid M, one can construct various algebraic objects possessing useful properties that can be used to answer purely combinatorial questions. Ardila and Klivans give one such example of an object in [AK06]. Given a loopless matroid M, they define a fan Σ_M, called the *Bergman* fan of M whose Chow ring A_M leads to many interesting combinatorial results [Ard24]. Most notably, Chow rings of Bergman fan of matroids are used in Adiprasito's, Huh's, and Katz's proof of the Heron-Rota-Welsh conjecture which asserts the absolute values of the characteristic polynomial of any matroid are log-concave [AHK18]. Ardila, Denham, and Huh give another interesting class of fans in [ADH23]. Given a loopless and coloopless M, they define a fan $\Sigma_{M,M^\perp}$, called the *conormal* fan of M, and they use its Chow ring $A_{M,M^\perp}$ to prove the log concavity of the h-vector of the broken circuit complex.

A lot remains unknown about Chow rings of conormal fans relative to Chow rings of Bergman fans. Our main point of comparison will be the Hilbert-Poincaré series of the Chow rings of these two classes of fans. For a graded ring R, let $H_R(t)$ denote its Hilbert-Poincaré series over $\mathbb{Z}$. The main result of [AHK18] about the validity of the "Kähler" package for the Chow ring A_M of the Bergman fan Σ_M directly implies that the coefficients of $H_{A_M}(t)$ are palindromic and form a unimodal sequence [Fer+24, p. 12-13]. Ferroni, Matherne, Stevens and Vecchi in [Fer+24] further investigate the Hilbert-Poincaré series of the Chow ring of Bergman fans with the goal of studying the following conjecture:

Conjecture 1.1. ([Gal05, Conjecture 8.18]) The Hilbert-Poincaré series $H_{A_M}(t)$ of the Chow ring A_M is real-rooted for any loopless matroid M.

In particular, they prove that $H_{A_M}(t)$ is γ-positive for any loopless matroid M [Fer+24, Theorem 3.25]. For the definition of γ-positivity, we refer the reader to Section 2. In addition, they prove the following general formula for $H_{A_M}(t)$, when M is an arbitrary loopless matroid, in terms of chains of flats of M:

Proposition 1.2. ([Fer+24, Proposition 3.15]) For an arbitrary loopless matroid M,

$$H_{A_M}(t) = \sum_{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m} \prod_{i=1}^{m} \frac{t(1 - t^{\operatorname{rank}(F_i) - \operatorname{rank}(F_{i-1}) - 1})}{1 - t}.$$

Here, the sum is taken over all the nonempty chains of flats of M starting at the empty set, i.e., $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m$. This formula is a direct consequence of the explicit Gröbner basis for the Chow ring of Bergman fans provided by Feichtner and Yuzvinsky in [FY04]. In addition, when M is a loopless uniform matroid of size n and rank r, Ferroni, Matherne, Stevens and Vecchi prove an explicit formula for $H_{A_M}(t)$:

Proposition 1.3. ([Fer+24, Theorem 3.25]) For a loopless uniform matroid M of size n and rank r,

$$H_{A_M}(t) = \sum_{j=0}^{r-1} \binom{n}{j} d_j(t)(1 + t + \cdots + t^{r-1-j})$$

where $d_j(x)$ denotes the j-th derangement polynomial.

Brändén and Vecchi further prove that $H_{A_M}(t)$ is real-rooted when M is uniform in [BV25, Theorem 3.1].

In contrast, there has been no discussion about the palindromicity, unimodality, γ-positivity or real-rootedness of the Hilbert-Poincaré series of conormal fans in the literature yet, nor has there been explicit computations of of this series for any special families of matroids. In this paper, we aim to fill this vacuum by partially answering the following question:

Question 1.4. Is the Hilbert-Poincaré series $H_{A_{M,M^\perp}}(t)$ of a loopless and coloopless matroid M palindromic, real-rooted, γ-positive or unimodal? And can we compute it explicitly for special families of matroids?

In particular, as our main result, we show that the conormal Chow ring $A_{M,M^\perp}$ is isomorphic to a tensor product of Chow rings $A_M, A_{M^\perp}$.

Proposition 1.5. *Let M be a loopless and coloopless uniform matroid. Then we have*

$$A_{M,M^\perp} \cong A_M \otimes_{\mathbb{Z}} A_{M^\perp}.$$

As a direct corollary of Proposition 1.5, we get that the Hilbert-Poincaré series $H_{A_{M,M^\perp}}(t)$ is a product of the Hilbert-Poincaré series of $A_M, A_{M^\perp}$.

Corollary 1.6. *If M is a loopless uniform matroid, then*

$$H_{A_{M,M^\perp}}(t) = H_{A_M}(t) H_{A_{M^\perp}}(t).$$

Combining Corollary 1.6 with Proposition 1.3, we get the following explicit formula for the Hilbert-Poincaré series $H_{A_{M,M^\perp}}(t)$ when M is uniform:

Proposition 1.7. *Let M be a uniform matroid of size n and rank r. Then*

$$H_{A_{M,M^\perp}}(t) = \left(\sum_{j=0}^{r-1} \binom{n}{j} d_j(t)(1 + t + \cdots + t^{r-1-j}) \right) \left(\sum_{j=0}^{n-r-1} \binom{n}{j} d_j(t)(1 + t + \cdots + t^{n-r-1-j}) \right).$$

In addition, by combining 1.6 with Theorem 3.1 in [BV25], we get the following proposition about the real-rootedness, γ-positivity and unimodality of $H_{A_{M,M^\perp}}(t)$:

Proposition 1.8. *Let M be a uniform matroid. Then $H_{A_{M,M^\perp}}(t)$ is real-rooted, γ-positive, and unimodal.*

The conormal Chow ring $A_{M,M^\perp}$ of a matroid M is defined in terms of the *biflats*, *bichains* and *biflags* of M. These three notions are analogues of the notions of flats, chains and flags of M, respectively, and will be defined (together with conormal Chow rings) later in Section 2. In Proposition 3.1 and 3.3, we prove two important properties about the structure of biflats, bichains and biflags of a uniform matroid M. These two propositions are crucial for our proof of Proposition 1.5. In fact, in section 5, we discuss how if any matroid M satisfies the statements of Proposition 3.1 and Proposition 3.3, then it also satisfies the statement of Propositon 1.5. On the other hand, in Proposition 5.1 and Proposition 5.4, we prove that if a matroid M satisfies the statement of Proposition 3.1 or Proposition 3.3, then M must be uniform. We refrain from stating the precise statments of Proposition 3.1 and Proposition 3.3 for now, and postpone it to Section 3.

The outline of the paper is as follows: In Section 2, we review the definitions of Bergman and conormal fans and their Chow rings as well as the definitions of palindromicity, unimodality, and γ-positivity. We also define derangement polynomials and give a sketch of the proof of the derangement polynomial formula for the Hilbert-Poincaré series of Chow rings of uniform matroids (Proposition 1.3) described in [Fer+24]. In Section 3, we prove Proposition 3.1 and 3.3. In Section 4, we prove Proposition 1.5, Corollary 1.6, Proposition 1.7, and Proposition 1.8. In Section 5, we prove Proposition 5.1 and Proposition 5.4. In Section 6, we discuss potential further research questions.

2 Background

2.1 Bergman and Conormal Fans and their Chow Rings

We now define Bergman and conormal fans of arbitrary matroids and their respective Chow rings. We rely on definitions of the fans given in [NP24], and the definitions of the Chow rings given in [ADH23] and [Ard24]. For the remainder of this section, let $M = (E, \mathbf{F}_M)$ be a loopless matroid with a ground set E and a set of flats $\mathbf{F}_M$. Recall that we can define a poset structure on $\mathbf{F}_M$ by letting $F \leq F'$ whenever $F \subseteq F'$ for any two flats $F, F' \in \mathbf{F}_M$. Let $M^\perp = (E, \mathbf{F}_{M^\perp})$ be the dual matroid of M. Let U_n^r denote the uniform matroid of rank r with ground set $\{1, 2, ..., n\}$, and recall that the dual of U_n^r is the uniform matroid U_n^{n-r}.

The Bergman fan Σ_M of the matroid M is the simplicial fan inside the *tropical projective space* N_E whose cones corresponds to *flags of flats* of M.

Definition 2.1. Given a ground set E, the *tropical projective space* N_E is the $(n-1)$-dimensional real vector space

$$\mathbb{R}^E/\mathbb{R}\mathbf{e}_E, \quad \mathbf{e}_E = \sum_{i \in E} e_i$$

where e_i denotes the i-th standard basis vector of $\mathbb{R}^E$.

The support of the Bergman fan Σ_M is *tropical linear space* $\mathrm{trop}(M)$ of M.

Definition 2.2. The *tropical linear space* of a matroid M is defined to be

$$\mathrm{trop}(M) := \{z \in N_E : \min_{i \in C} z_i \text{ is achieved at least twice for every circuit } C \text{ of } M\} \subseteq N_E.$$

The Bergman fan Σ_M is a subdivision of $\mathrm{trop}(M)$ defined in the following way:

Definition 2.3. The *Bergman fan* Σ_M of the matroid M is the simplicial fan in N_E whose support is $\mathrm{trop}(M)$ and whose cones are

$$\sigma_{\mathcal{F}} = \mathrm{cone}(\mathbf{e}_F)_{F \in \mathcal{F}}, \text{ for any flag of flats } \mathcal{F} \text{ of } M$$

where a *flag of flats* $\mathcal{F}$ is any subset $\{F_1, ..., F_m\}$ of the set of flats $\mathbf{F}_M$ such that $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_m$.

Definition 2.4. The *Chow ring* of the matroid M is the graded ring

$$A_M := S_M/(I_M + J_M)$$

where

- $S_M := \mathbb{Z}[x_F : F \in \mathbf{F}_M \setminus \{\emptyset, E\}]$,
- $I_M := \langle x_F x_{F'} : x_F, x_{F'} \in \mathbf{F}_M \setminus \{\emptyset, E\}$ are incomparable$\rangle$,
- $J_M := \langle \gamma_i - \gamma_j : i, j \in E \rangle$ where
$$\gamma_i := \sum_{F \ni i} x_F.$$

The Hilbert-Poincaré series $H_{A_M}(t)$ of M is sometimes referred to as the *Chow polynomial* of M.

Example 2.5. We compute the Chow ring A_M of the uniform matroid $M = U_4^2$. We first compute each one of S_M, I_M, J_m individually. We have $F_M = \{\emptyset, 1, 2, 3, 4, E\}$, so we get

$$S_M = \mathbb{Z}[x_1, x_2, x_3, x_4].$$

In addition, all pairs of nonempty proper flats of M are incomparable, so I_M is generated by monomials $x_i x_j$ where $i, j \in \{1, 2, 3, 4\}$. Lastly, for any $i \in E$, we have

$$\sum_{F \ni i} x_F = x_i.$$

This implies that J is generated by monomials $x_i - x_j$ for all $i, j \in \{1, 2, 3, 4\}$. Thus,

$$A_M = S_M/(I_M + J_M) \cong \mathbb{Z}[x_1]/(x_1^2).$$

Hence, $H_{A_M}(t) = 1 + t$.

Now we describe conormal fans and their Chow rings. For the remainder of the section, assume that M is a loopless and coloopless matroid. The conormal fan of the matroid M is the simplicial fan inside $N_E \times N_E$ whose support is $\text{trop}(M) \times \text{trop}(M^\perp)$ and whose cones corresponds *biflags of biflats.*

Definition 2.6. A *biflat* $F|G$ consists of a flat $F \in \mathbf{F}_M$ and a dual flat $G \in \mathbf{F}_{M^\perp}$ such that both are nonempty, at least one of them is proper and,

$$F \cup G = E.$$

The set of biflats of M is denoted as $\mathbf{BF}_{M,M^\perp}$. We endow the set of biflats of a matroid with the structure of a poset by defining $F|G \leq F'|G'$ if and only if $F \subseteq F'$ and $G \supseteq G'$. A chain of the form $F_1|G_1 \leq \cdots \leq F_n|G_n$ where all $F_i|G_i$'s are distinct is called a *bichain* of M. An important class of bichains is given in the following definition:

Definition 2.7. A bichain $\mathcal{F}|\mathcal{G} = F_1|G_1 \leq \cdots \leq F_n|G_n$ of M is called a *biflag* if and only if we have

$$\bigcup_{F|G \in \mathcal{F}|\mathcal{G}} F \cap G \neq E.$$

Example 2.8. Let $M = U_4^2$. The set of biflats of M is given by

$$\mathbf{BF}_{M,M^\perp} = \{1|E, 2|E, 3|E, 4|E, E|1, E|2, E|3, E|4\}$$

The set of bichains of M consists of 1-element bichains $F|G$ where $F|G \in \mathbf{BF}_{M,M^\perp}$ as well as the collection of 2-element bichains $F|E \leq E|G$ where $F \in \mathbf{F}_M \setminus \{\emptyset, E\}$ and $G \in F_{M^\perp} \setminus \{\emptyset, E\}$. One can verify that each of these bichains is also a biflag of M, so the set of bichains and biflags of M is equal.

For any subset $S \subseteq E$, define the vectors $\mathbf{e}_S$ and $\mathbf{f}_S$ to be the image of $\sum_{i \in S} \mathbf{e}_i$ in the first and the second copy of N_E, respectively. For two subsets $S, T \subseteq E$, define $\mathbf{e}_{S|T}$ to be the vector $(\mathbf{e}_S, \mathbf{f}_T)$ in $N_E \times N_E$.

Definition 2.9. The *conormal fan* of M, denoted $\Sigma_{M,M^\perp}$, is the simplicial fan in $N_E \times N_E$ whose support is $\mathrm{trop}(M) \times \mathrm{trop}(M^\perp)$, and whose cones are given by

$$\sigma_{\mathcal{F}|\mathcal{G}} = \mathrm{cone}\{\mathbf{e}_{F|G}\}_{F|G \in \mathcal{F}|\mathcal{G}}, \text{ for any biflag of biflats } \mathcal{F}|\mathcal{G}.$$

Remark 2.10. The definition of the tropical linear space $\mathrm{trop}(M)$ reveals why we require our matroid M to be a loopless and coloopless. If M contains a loop a, then $\{a\}$ is a circuit and hence

$$\mathrm{trop}(M) \subseteq \{z \in N_E : \min_{i \in \{a\}} z_i \text{ is achieved at least twice}\} = \emptyset.$$

Similarly, we get $\mathrm{trop}(M^\perp) = \emptyset$ if M has a coloop.

Definition 2.11. The c*onormal Chow ring* of M is the graded ring

$$A_{M,M^\perp} := S_{M,M^\perp}/(I_{M,M^\perp} + J_{M,M^\perp})$$

where

- $S_{M,M^\perp} := \mathbb{Z}[x_{F|G} : F|G \in \mathbf{BF}_{M,M^\perp}]$,
- $I_{M,M^\perp} := \langle x_{\mathcal{F}|\mathcal{G}} : \mathcal{F}|\mathcal{G}$ is a subset of $\mathbf{BF}_{M,M^\perp}$ that is not a biflag$\rangle$ where $x_{\mathcal{F}|\mathcal{G}}$ is a monomial defined by
$$x_{\mathcal{F}|\mathcal{G}} := \prod_{F|G \in \mathcal{F}|\mathcal{G}} x_{F|G},$$
- $J_{M,M^\perp} = \langle \gamma_i - \gamma_j : i,j \in E\rangle + \langle \overline{\gamma}_i - \overline{\gamma}_j : i,j \in E\rangle$ where $\gamma_i, \overline{\gamma}_i$ are linear functions defined by
$$\gamma_i := \sum_{F \ni i, F \neq E} x_{F|G}, \quad \overline{\gamma}_i := \sum_{G \ni i, G \neq E} x_{F|G}.$$

Example 2.12. Let $M = U_4^2$. Based on Example 2.8, we get

$$S_{M,M^\perp} = \mathbb{Z}[x_{1|E}, x_{2|E}, x_{3|E}, x_{4|E}, x_{E|1}, x_{E|2}, x_{E|3}, x_{E|4}]$$

In addition, since every bichain of M is a biflag, then one can check that

$$I_{M,M^\perp} = \langle x_{i|E}x_{j|E} : i,j \in E, i \neq j\rangle + \langle x_{E|i}x_{E|j} : i,j \in E, i \neq j\rangle.$$

Finally, for any $i \in E$,

$$\gamma_i = \sum_{F \ni i, F \neq E} x_{F|G} = x_{i|E}, \qquad \overline{\gamma}_i = \sum_{G \ni i, G \neq E} x_{F|G} = x_{E|i}.$$

Thus,
$$J_{M,M^\perp} = \langle x_{i|E} - x_{j|E} : i,j \in E\rangle + \langle x_{E|i} - x_{E|j} : i,j \in E\rangle.$$

Now, it is straightforward to check that

$$A_{M,M^\perp} = S_{M,M^\perp}/(I_{M,M^\perp} + J_{M,M^\perp}) \cong \mathbb{Z}[x_{1|E}, x_{E|1}]/(x_{1|E}^2, x_{E|1}^2).$$

Hence, $H_{A_{M,M^\perp}}(t) = 1 + 2t + t^2$.

2.2 Unimodality, γ-Positivity and Real-Rootedness

We define three conditions on polynomials that are relevant for our study of the Hilbert-Poincaré series $H_{A_{M,M^\perp}}(t)$ and $H_{A_M}(t)$, namely unimodality, γ-positivity and real-rootedness. A polynomial $f(x) \in \mathbb{R}[x]$ is said to be real-rooted if all of its roots are real. As it is usually hard to prove that a polynomial has only real roots, it is useful to consider proving the unimodality and γ-positivity of a polynomial first.

Definition 2.13. A polynomial $f(x) = a_n x^n + \cdots + a_0$ is said to be *unimodal* if its coefficients form a unimodal sequence, i.e., there exists some $0 \leq t \leq n$ such that $a_0 \leq a_1 \leq \cdots \leq a_t$ and $a_t \geq a_{t+1} \geq \cdots \geq a_n$.

While the notion of unimodality applies to any polynomial, the concept of γ-positivity only applies to symmetric polynomials.

Definition 2.14. A polynomial $f(x) = \sum_i a_i x^i$ is said to be *symmetric* with center of symmetry $d/2$ if there exists a $d \in \mathbb{Z}$ such that $a_i = a_{d-i}$ for all $i \in \mathbb{Z}$. This condition is equivalent to the existence of a $d \in \mathbb{Z}$ such that $x^d f(x^{-1}) = f(x)$. If $d = \deg(f)$, then we call $f(x)$ *palindromic.*

Symmetric polynomials can be expressed in different bases. For example, the set $\{x^i(1+x)^{d-2i}\}_{i=0}^{\lfloor \frac{d}{2} \rfloor}$ forms a basis for the space of all symmetric polynomials with center of symmetry $\frac{d}{2}$. This is guaranteed by the following proposition whose proof can be found in [Gal05]:

Proposition 2.15. *If $f(x)$ is a symmetric polynomial with center of symmetry $\frac{d}{2}$, then there exist integers $\gamma_0, ..., \gamma_{\lfloor \frac{d}{2} \rfloor}$ such that*

$$f(x) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i x^i (1+x)^{d-2i}. \tag{1}$$

Definition 2.16. For a symmetric polynomial $f(x)$ with center of symmetry $\frac{d}{2}$, define $\gamma_0, ..., \gamma_{\lfloor d/2 \rfloor}$ as in (1). We define the *γ-polynomial associated to f* by

$$\gamma(f, x) := \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i x^i.$$

If $f(x)$ is palindromic, we say that $f(x)$ is *γ-positive* if all the coefficients of $\gamma(f,x)$ are nonnegative.

Example 2.17. Consider the palindromic $f(x) = 1+37x+72x^2+37x^3+x^4$ with center of of symmetry $d/2 = 2$. It is immediate that $f(x)$ is unimodal. To check γ-positivity, we express $f(x)$ in the basis $\{x^i(1+x)^{d-2i}\}_{i=0}^{\lfloor \frac{d}{2} \rfloor} = \{(1+x)^4, x(1+x)^2, x^2\}$ in the following way:

$$f(x) = (1+x)^4 + 33x(1+x)^2 + 0x^2.$$

Hence, $\gamma(f,x) = 1 + 33x$, and $f(x)$ is γ-positive.

Unimodality and γ-positivity are weaker condition on a polynomial than real-rootedness. More precisely, we have the following chain of implications whose proofs can be found in [Brä15] and [Gal05].

Proposition 2.18. *Let $f(x)$ be a symmetric polynomial with nonnegative coefficients. Then*

$$f(x) \text{ is real-rooted} \implies f(x) \text{ is } \gamma\text{-positive} \implies f(x) \text{ is unimodal.}$$

Example 2.19. The palindromic polynomial $g(x) = 1 + 5x + 6x^2 + 5x^3 + x^4$ is unimodal, but it is not γ-positive since $\gamma(g,x) = 1 + x - x^2$. The palindromic polynomial $h(x) = 1 + 4x + 6x^2 + 4x^3 + x^4$

is γ-positive since $\gamma(h,x) = 1$, but it is not real-rooted as all its roots are not real. The palindromic γ-positive polynomial $f(x) = 1 + 37x + 72x^2 + 37x^3 + x^4$ from Example 2.17 is real-rooted with roots $-1, -1, \frac{2}{-35-\sqrt{1221}}, \frac{-35-\sqrt{1221}}{2}$.

2.3 Derangement Polynomials and The Hilbert-Poincaré series of Chow Rings of Uniform Matroids

We define derangement polynomials and sketch the proof provided in [Fer+24] of Proposition 1.3 which provides an explicit formula involving derangement polynomials for the Hilbert-Poincaré series $H_{A_M}(t)$ when M is uniform. For the full proof, we refer the reader to [Fer+24].

We say that a permutation $\sigma \in S_n$ on $\{1, 2, ..., n\}$ is a derangement if $\sigma(i) \neq i$ for all i. We denote the set of all derangements of S_n as $\mathfrak{D}_n$. We define the *n-th derangement polynomial* $d_n(x)$ for $n \geq 1$ as

$$d_n(x) := \sum_{\sigma \in \mathfrak{D}_n} x^{\mathrm{exc}(\sigma)}$$

where $\mathrm{exc}(\sigma) := |\{i \in \{1, 2, ..., n\} : \sigma_i > i\}|$ denotes the *excedances* of σ. The first few values of $d_n(x)$ are as follows:

$$d_n(x) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n = 1 \\ x & \text{for } n = 2 \\ x^2 + x & \text{for } n = 3 \\ x^3 + 7x^2 + x & \text{for } n = 4 \\ x^4 + 21x^3 + 21x^2 + x & \text{for } n = 5 \\ \vdots \end{cases}$$

We note here that $d_0(x) = 1$ by convention. For a summary of the properties of derangement polynomials, we refer the reader to [Fer+24]. We replicate the statement of Proposition 1.3 here for convenience.

Proposition 2.20. [Fer+24, Theorem 3.25] For a loopless uniform matroid M of size n and rank r,

$$H_{A_M}(t) = \sum_{j=0}^{r-1} \binom{n}{j} d_j(t)(1 + t + \cdots + t^{r-1-j}).$$

where $d_j(x)$ denotes the j-th derangement polynomial.

Proof. (Sketch) The first step of the proof is to let $M = U_n^n$ and to prove that derangement polynomials satisfy the following equality:

$$d_n(x) = \sum_{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m = E} \prod_{i=1}^{m} \frac{x(1 - x^{\mathrm{rank}(F_i) - \mathrm{rank}(F_{i-1}) - 1})}{1 - x}.$$

Here, the sum is taken over all the nonempty chains of flats of M starting at the empty set and ending at E, i.e., $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m = E$. The way this equality is proved is by showing the right-hand side of the equality (which we denote as $\mathfrak{d}_n(x)$) satisfies the recurrence

$$\mathfrak{d}_n(x) = \sum_{j=0}^{n-2} \mathfrak{d}_j(x)(x + x^2 + \cdots + x^{n-j-1}),$$

which by [JMS19] implies that $d_n(x) = \mathfrak{d}_n(x)$ for $n \geq 1$. For the next step of the proof, assume $M = U_n^r$. By Proposition 1.2,

$$H_{A_M}(t) = \sum_{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m} \prod_{i=1}^{m} \frac{t(1 - t^{\operatorname{rank}(F_i) - \operatorname{rank}(F_{i-1}) - 1})}{1-t}.$$

By considering which chains of flats end in E or not, and by using the first step, we can rearrange the sum on the right in the following way:

$$\begin{aligned} H_{A_M}(t) &= \sum_{j=0}^{r-1} d_j(t) + \sum_{j=0}^{r-1} d_j(t) \frac{t(1-t^{r-j-1})}{1-t} \\ &= \sum_{j=0}^{r-1} d_j(t) \left(1 + \frac{t(1-t^{r-j-1})}{1-t}\right) \\ &= \sum_{j=0}^{r-1} d_j(t)(1 + t + \cdots + t^{r-1-j}). \end{aligned}$$

□

3 Structure of Biflats and Bichains of Uniform Matroids

We prove two useful facts for our computation of the conormal Chow ring $A_{M,M^\perp}$ when M is uniform in Section 4. In Proposition 3.1, we compute the biflats of the uniform matroid. In Proposition 3.3, we show that every bichain of the uniform matroid is a biflag. Recall the flats of the uniform matroid $M = U_n^r$ are given by set the $\{A \subseteq E : |A| \leq r-1\} \cup \{E\}$.

Proposition 3.1. *Let $M = U_n^r$ be the uniform matroid of rank r with ground set $E = \{1, ..., n\}$. Then*

$$\boldsymbol{BF}_{M,M^\perp} = \{F|E : F \in \boldsymbol{F}_M \backslash \{\emptyset, E\}\} \cup \{E|G : G \in \boldsymbol{F}_{M^\perp} \backslash \{\emptyset, E\}\}.$$

Equivalently,

$$\boldsymbol{BF}_{M,M^\perp} = \{F|E : F \subseteq E \text{ and } 1 \leq |F| \leq r-1\} \cup \{E|G : G \subseteq E \text{ and } 1 \leq |G| \leq n-r-1\}.$$

Proof. We first prove the reverse inclusion. The dual of U_n^r is U_n^{n-r}, hence

$$\mathbf{F}_{U_n^r} = \{F \subseteq E : |F| \leq r-1\} \cup \{E\} \text{ and } \mathbf{F}_{U_n^{n-r}} = \{G \subseteq E : |G| \leq n-r-1\} \cup \{E\}.$$

Thus, any $F|E$ in $\{F|E : F \subseteq E \text{ and } 1 \leq |F| \leq r-1\}$ is a biflat. Similarly, any $E|G$ in $\{E|G : G \subseteq E \text{ and } 1 \leq |G| \leq n-r-1\}$ is also a biflat, so the reverse inclusion holds.

For the forward inclusion, let $F|G \in \mathbf{BF}_{M,M^\perp}$. Then,

$$F \in \{A \subseteq E : |A| \leq r-1\} \cup \{E\} \text{ and } G \in \{A \subseteq E : |A| \leq n-r-1\} \cup \{E\}.$$

We know that at least one of F or G is proper. Without loss of generality, assume that $F \neq E$ so that $|F| \leq r-1$. We claim that $G = E$. Towards a contradiction, assume otherwise. Then, $G \in \{A \subseteq E : |A| \leq n-r-1\}$, and so $|G| \leq n-r-1$. By assumption, $F \cup G = E$, so $|F| + |G| \geq |E| = n$. On the other hand,

$$|F| + |G| \leq (r-1) + (n-r-1) = n-2 < n.$$

which is a contradiction. Therefore, $G = E$, and so

$$F|G \in \{F|E : F \subseteq E \text{ and } 1 \leq |F| \leq r-1\}.$$

If we assume that $G \neq E$, then with similar reasoning, we can show that

$$F|G \in \{E|A : A \subseteq E \text{ and } 1 \leq |A| \leq n-r-1\}.$$

□

Lemma 3.2. *Let $\mathcal{F}|\mathcal{G}$ be bichain of the uniform matroid $M = U_n^r$. Then*

$$\mathcal{F}|\mathcal{G} = F_1|E \leq F_2|E \leq \cdots \leq F_{j-1}|E \leq E|G_j \leq E|G_{j+1} \leq \cdots \leq E|G_n.$$

Proof. Let $\mathcal{F}|\mathcal{G} = F_1|G_1 \leq \cdots F_n|G_n$ be a bichain of M. By Proposition 3.1, any $F_i|G_i$ has the form $F_i|E$ or $E|G_i$. Suppose first that $G_i = E$ for each i. Then, $\mathcal{F}|\mathcal{G} = F_1|E \leq \cdots \leq F_n|E$, giving us the claim when $j = n+1$. Otherwise, assume that there is some j such that $G_j \neq E$, and assume that j is the smallest such index. If $j = 1$, then $G_i \neq E$ for each i since $G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n$. By Proposition 3.1, this forces $F_i = E$ for each i, and so $\mathcal{F}|\mathcal{G} = E|G_1 \leq \cdots \leq E|G_n$, giving us the third case. Otherwise, assume that $j > 1$. By the minimality assumption on j, we know that $G_1 = \cdots = G_{j-1} = E$. In addition, we get that $G_j, G_{j+1}, ..., G_n$ are all not equal to E since $G_j \supseteq G_{j+1} \supseteq \cdots \supseteq G_n$. Thus, $F_j = F_{j+1} = \cdots = F_n = E$ by Proposition 3.1. In total,

$$\mathcal{F}|\mathcal{G} = F_1|E \leq \cdots \leq F_{j-1}|E \leq E|G_j \leq \cdots \leq E|G_n.$$

□

Proposition 3.3. *Every bichain of the uniform matroid $M = U_n^r$ is a biflag.*

Proof. Let $\mathcal{F}|\mathcal{G}$ be a bichain of M. By Lemma 3.2, we get

$$\mathcal{F}|\mathcal{G} = F_1|E \leq F_2|E \leq \cdots \leq F_{j-1}|E \leq E|G_j \leq E|G_{j+1} \leq \cdots \leq E|G_n.$$

Suppose first $1 \leq j \leq n$. By Proposition 3.4, $|F_i| \leq r-1$ for all $1 \leq i \leq j-1$, and $|G_i| \leq n-r-1$ for all $j \leq i \leq n$. We know that $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{j-1}$. This implies that

$$\bigcup_{1 \leq i \leq j-1} F_i \cap E = F_{j-1} \cap E,$$

and hence,

$$|\bigcup_{1 \leq i \leq a} F_i \cap E| = |F_a \cap E| \leq r-1.$$

Since $G_j \supseteq G_{j+1} \supseteq \cdots \supseteq G_n$, then with similar reasoning, we have

$$|\bigcup_{j \leq i \leq n} E \cap G_i| = |E \cap G_b| \leq n-r-1.$$

In total,

$$\begin{aligned} |\bigcup_{F|G \in \mathcal{F}|\mathcal{G}} F \cap G| &\leq |\bigcup_{1 \leq i \leq a} F_i \cap E| + |\bigcup_{1 \leq i \leq b} E \cap G_i| \\ &\leq (r-1) + (n-r-1) = n-2 < n. \end{aligned}$$

Thus,

$$\bigcup_{F|G \in \mathcal{F}|\mathcal{G}} F \cap G \neq E,$$

showing that $\mathcal{F}|\mathcal{G}$ is a biflag. If $j = 0$, then we can follow a similar argument to the above case to conclude

$$|\bigcup_{F|G \in \mathcal{F}|\mathcal{G}} F \cap G| = |F_n \cap E| \leq r - 1 < n.$$

Similarly, if $j = n + 1$, then we can conclude that

$$|\bigcup_{F|G \in \mathcal{F}|\mathcal{G}} F \cap G| = |G_1 \cap E| \leq n - r - 1 < n.$$

In both cases, we get

$$\bigcup_{F|G \in \mathcal{F}|\mathcal{G}} F \cap G \neq E.$$

□

4 The Conormal Chow Ring for Uniform Matroids

We express the conormal Chow ring $A_{M,M^\perp}$ of a loopless and coloopless uniform matroid M as a tensor product of the Chow rings A_M and $A_{M^\perp}$ of M and $M^\perp$, respectively (Proposition 1.5). The definition of $A_{M,M^\perp}$ includes three components, namely $S_{M,M^\perp}, I_{M,M^\perp}, J_{M,M^\perp}$, as defined in Definition 2.11. In Lemma 4.2, we express $S_{M,M^\perp}$ as a tensor product of two polynomials rings, S_1 and S_2 with $S_1 \cong S_M$ and $S_2 \cong S_{M^\perp}$. In Lemma 4.3, we express $I_{M,M^\perp}$ as a sum of ideal extensions $(I_1)^e + (I_2)^e$ where $I_1 \cong I_M$ and $I_2 \cong I_{M^\perp}$. Similarly, in Lemma 4.4, we express $J_{M,M^\perp}$ as a sum of ideal extensions $(J_1)^e + (J_2)^e$ where $J_1 \cong J_M$ and $J_2 \cong J_{M^\perp}$. By substituting the isomorphisms in these three lemmas to Lemma 4.1, we deduce Proposition 1.5. Crucially, in our computation of $I_{M,M^\perp}$, we use the fact that all bichains of a uniform matroid are biflags (Lemma 3.3) to deduce that is is a Stanley-Reisner ideal. This allows us to find the generators of $I_{M,M^\perp}$ by finding incomparable biflats of M.

Lemma 4.1. *Let k be commutative ring. For commutative k-algebras R, S and ideals $I \subseteq R, J \subseteq S$, we have*

$$(R \otimes_k S)/(I^e + J^e) \cong R/I \otimes_k S/J$$

where I^e denotes the extension of I along $R \to R \otimes_k S$, and similarly for J^e.

Proof. We have a natural homomorphism $R \otimes_k S \xrightarrow{\phi} R/I \otimes_k S/J$ that sends $x \otimes y \mapsto [x] \otimes_k [y]$ where $[x]$ denotes the equivalence class of x in R/I and similarly for $[y]$. This map is surjective since for any pure tensor $[x] \otimes_k [y]$ in $R/I \otimes_k S/J$, $\phi(x \otimes_k y) = [x] \otimes_k [y]$, and so surjectivity holds for any arbitrary element of $R/I \otimes_k S/J$. The isomorphism thus follows from that the fact that $\ker(\phi) = I^e + J^e$. □

Lemma 4.2. *Let $M = U_n^r$ be the uniform matroid, and let $S_1 = \mathbb{Z}[x_{F|E} : F \in \boldsymbol{F}_M \backslash \{\emptyset, E\}]$ and $S_2 = \mathbb{Z}[x_{E|F} : F \in \boldsymbol{F}_{M^\perp} \backslash \{\emptyset, E\}]$. Then*

$$S_{M,M^\perp} \cong S_1 \otimes_{\mathbb{Z}} S_2.$$

Proof. This immediately follows from our result in Proposition 3.1 that

$$\begin{aligned}\mathbf{BF}_{M,M^\perp} &= \{F|E : F \subseteq E \text{ and } 1 \leq |F| \leq r-1\} \cup \{E|G : G \subseteq E \text{ and } 1 \leq |G| \leq n-r-1\} \\ &= \{F|E : F \in \mathbf{F}_M \backslash \{\emptyset, E\}\} \cup \{E|G : G \in \mathbf{F}_{M^\perp} \backslash \{\emptyset, E\}\}.\end{aligned}$$

□

Lemma 4.3. *Let $M = U_n^r$, and suppose that S_1 and S_2 are defined as in the statement of Lemma 4.2. In S_1, define I_1 as the ideal generated by $x_{F|E}x_{F'|E}$ where $F, F' \in \boldsymbol{F}_M \backslash \{\emptyset, E\}$ are incomparable. Similarly, in S_2, define I_2 as the ideal generated by $x_{E|G}x_{E|G'}$ where $G, G' \in \boldsymbol{F}_{M^\perp} \backslash \{\emptyset, E\}$ are incomparable. Then*

$$I_{M,M^\perp} = (I_1)^e + (I_2)^e$$

where $(I_i)^e$ denotes the extension of I_i along the inclusion map $S_i \to S_1 \otimes_{\mathbb{Z}} S_2$ for $i = 1, 2$.

Proof. We proved in Proposition 3.3 that every bichain of M is a biflag, so $I_{M,M^\perp}$ is generated by $x_{\mathcal{F}|\mathcal{G}}$ where $\mathcal{F}|\mathcal{G}$ is not a bichain. Thus, $I_{M,M^\perp}$ is equal to a Stanley-Resiner ideal I_Δ where Δ is the simplicial complex where the vertices are biflats of M and the faces are bichains of M, and so $I_{M,M^\perp}$ is generated by $x_{F|G}x_{F'|G'}$ where $F|G, F'|G'$ are incomparable biflats. We examine which biflats of M are incomparable. If $F|G, F'|G' \in \mathbf{BF}_{M,M^\perp}$ are not comparable, then we have two possible cases: The first case is $F|G = F|E$ and $F'|G' = F'|E$, and the second case is $F|G = E|G$ and $F'|G' = E|G'$. Therefore,

$$I_{M,M^\perp} = I_1' + I_2'$$

where I_1' is the ideal in $S_{M,M^\perp}$ generated by $x_{F|E}x_{F'|E}$ where $F', F \in \mathbf{F}_M \backslash \{\emptyset, E\}$ are incomparable, and I_2' is the ideal in $S_{M,M^\perp}$ generated by $x_{E|G}x_{E|G'}$ where $G, G' \in \mathbf{F}_{M^\perp} \backslash \{\emptyset, E\}$ are incomparable. It is quick to verify that

$$(I_1)^e = I_1', \quad (I_2)^e = I_2'$$

where the extension is along the inclusion maps $S_1 \to S_{M,M^\perp}, S_2 \to S_{M,M^\perp}$ and so we have

$$I_{M,M^\perp} = (I_1)^e + (I_2)^e.$$

□

Lemma 4.4. *Let $M = U_n^r$, and suppose that S_1 and S_2 are defined as in the statement of Lemma 4.2. In S_1, define J_1 as the ideal generated by $\gamma_i - \gamma_j$ for any $i, j \in E$. Similarly, in S_2, define J_2 as the ideal generated by $\overline{\gamma}_i - \overline{\gamma}_j$ for any $i, j \in E$. Then*

$$J_{M,M^\perp} = (J_1)^e + (J_2)^e$$

where $(J_i)^e$ denotes the extension of J_i along the inclusion map $S_i \to S_1 \otimes_{\mathbb{Z}} S_2$ for $i = 1, 2$.

Proof. We have that

$$J_{M,M^\perp} = J_1' + J_2'$$

where J_1' is the ideal generated by $\gamma_i - \gamma_j$ for any $i, j \in E$ and J_2' is the ideal generated by $\overline{\gamma}_i - \overline{\gamma}_j$ for any $i, j \in E$. It is quick to verify that

$$(J_1)^e = J_1', (J_2)^e = J_2'$$

where the extension is along the inclusion maps $S_1 \to S_{M,M^\perp}, S_2 \to S_{M,M^\perp}$. Thus,

$$J_{M,M^\perp} = (J_1)^e + (J_2)^e.$$

□

Proposition 4.5 (Proposition 1.5)**.** *For $M = U_n^r$, we have*

$$A_{M,M^\perp} \cong A_M \otimes_{\mathbb{Z}} A_{M^\perp}.$$

Proof. We have

$$A_{M,M^\perp} = S_{M,M^\perp}/(I_{M,M^\perp} + J_{M,M^\perp})$$

where the definition of $S_{M,M^\perp}, I_{M,M^\perp}, J_{M,M^\perp}$ is given in Definition 2.11. Define $S_1, S_2, I_1, I_2, J_1, J_2$ as in the statements of Lemmas 4.2, 4.3 and 4.4. By these three lemmas, we get

$$\begin{aligned} A_{M,M^\perp} &= S_{M,M^\perp}/(I_{M,M^\perp} + J_{M,M^\perp}) \\ &\cong (S_1 \otimes_{\mathbb{Z}} S_2)/((I_1)^e + (I_2)^e + (J_1)^e + (J_2)^e) \\ &\cong (S_1 \otimes_{\mathbb{Z}} S_2)/((I_1 + J_1)^e + (I_2 + J_2)^e) \\ &\cong (S_1/I_1 + J_1) \otimes_{\mathbb{Z}} (S_2/(I_1 + J_2)) \end{aligned}$$

where the last isomorphism holds by Lemma 4.1. Now, one can verify, through the isomorphism $x_{F|E} \mapsto x_F$, that

$$S_1/(I_1 + J_1) \cong S_M/(I_M + J_M) = A_M.$$

Similarly, through the isomorphism $x_{E|G} \mapsto x_G$, one can verify that

$$S_2/(I_2 + J_2) \cong S_{M^\perp}/(I_{M^\perp} + J_{M^\perp}) = A_{M^\perp}.$$

In total, we get

$$A_{M,M^\perp} \cong (S_1/I_1 + J_1) \otimes_{\mathbb{Z}} (S_2/(I_1 + J_2)) \cong A_M \otimes_{\mathbb{Z}} A_{M^\perp}.$$

□

Corollary 4.6 (Proposition 1.6)**.** *Let $M = U_n^r$. Then,*

$$H_{A_{M,M^\perp}}(t) = H_{A_M}(t) H_{A_{M^\perp}}(t).$$

Proof. By Proposition 4.5, we get $A_{M,M^\perp} \cong A_M \otimes_{\mathbb{Z}} A_{M^\perp}$. Thus, the claim follows from the multiplicative property of Hilbert-Poincaré series over tensor products. □

Proposition 4.7 (Proposition 1.7)**.** *Let $M = U_n^r$. Then we have*

$$H_{A_{M,M^\perp}}(t) = \left(\sum_{j=0}^{r-1} \binom{n}{j} d_j(t)(1 + t + \cdots + t^{r-1-j}) \right) \left(\sum_{j=0}^{n-r-1} \binom{n}{j} d_j(t)(1 + t + \cdots + t^{n-r-1-j}) \right).$$

Proof. This follows immediately from Corollary 4.6 and Proposition 1.3. □

Proposition 4.8 (Proposition 1.8)**.** *Let $M = U_n^r$. Then, $H_{A_{M,M^\perp}}$ is real-rooted, γ-positive, and unimodal.*

Proof. The Hilbert-Poincaré series $H_{A_{M,M^\perp}}(t)$ is real-rooted since $H_{A_{M,M^\perp}}(t) = H_{A_M}(t) H_{A_{M^\perp}}(t)$ by Corollary 4.6, and by Proposition [BV25], each one of $H_{A_M}(t), H_{A_{M^\perp}}(t)$ is real-rooted. Moreover, $H_{A_{M,M^\perp}}(t)$ is palindromic since it is the product of two palindromic polynomials $H_{A_M}(t), H_{A_{M^\perp}}(t)$, and hence, $H_{A_{M,M^\perp}}(t)$ is γ-positive and unimodal. □

In the table below, we compute $H_{A_{M,M^\perp}}(t)$ for $M = U_n^r$ where $1 \le r \le n-1$ and $r = 2, 3, ..., 7$. We note that when M is uniform, by Corollary 4.6,

$$H_{A_{M,M^\perp}}(t) = H_{A_M}(t) H_{A_{M^\perp}}(t) = H_{A_{M^\perp}}(t) H_{A_{(M^\perp)^\perp}}(t) = H_{A_{M^\perp,(M^\perp)^\perp}}(t).$$

Hence the value of the Hilbert-Poincare series of the conormal Chow ring of a matroid M and its dual $M^\perp$ is the same when M is uniform. This fact is highlighted in the left column of the table.

Table 1: Examples of $H_{A_M}(t)$ for some uniform matroids M.

M	$H_{A_{M,M^\perp}}(t)$
U_2^1	1
U_3^1, U_3^2	$1+t$
U_4^1, U_4^3	$1+7t+t^2$
U_4^2	$1+2t+t^2$
U_5^1, U_5^4	$1+21t+21t^2+t^3$
U_5^2, U_5^3	$1+12t+12t^2+t^3$
U_6^1, U_6^5	$1+51t+16t^2+51t^3+t^4$
U_6^2, U_6^4	$1+37t+72t^2+37t^3+t^4$
U_6^3	$1+32t+258t^2+32t^3+t^4$
U_7^1, U_7^6	$1+113t+813t^2+813t^3+113t^4+t^5$
U_7^2, U_7^5	$1+93t+429t^2+429t^3+93t^4+t^5$
U_7^3, U_7^4	$1+79t+1312t^2+1312t^3+79t^4+t^5$

5 Uniqueness of the Uniform Case Method

The proof of Proposition 1.5 crucially relies on Propositions 3.1 and 3.3, so one might hope if the latter two propositions holds true in the case of any other matroids. Namely, we would like to investigate if there are any non-uniform matroids M such that

$$\mathbf{BF}_{M,M^\perp} = \{F|E : F \in \mathbf{F}_M \backslash \{\emptyset, E\}\} \cup \{E|G : G \in \mathbf{F}_{M^\perp} \backslash \{\emptyset, E\}\},$$

or if there any non-uniform matroids M such that every bichain is a biflag. If a matroid M exists such that these two properties are satisfied, then one can imitate the proofs of Lemmas 4.2, 4.3, 4.4 and Proposition 4.5 that we described in Section 4 to prove that $A_{M,M^\perp} \cong A_M \otimes_{\mathbb{Z}} A_{M^\perp}$. In the following discussion, we prove that these two properties are only true in the case of uniform matroids.

Proposition 5.1. *Suppose that M is a loopless and coloopless matroid such that*

$$\boldsymbol{BF}_{M,M^\perp} = \{F|E : F \in \boldsymbol{F}_M \backslash \{\emptyset, E\}\} \cup \{E|G : G \in \boldsymbol{F}_{M^\perp} \backslash \{\emptyset, E\}\}.$$

Then, M is uniform.

Proof. We can follow the same steps of the proof of Lemma 3.2 to deduce that

$$\mathcal{F}|\mathcal{G} = F_1|E \leq F_2|E \leq \cdots \leq F_{j-1}|E \leq E|G_j \leq E|G_{j+1} \leq \cdots \leq E|G_n.$$

Then the claim follows from [NP24]. □

Lemma 5.2. *Let F be a proper flat of a loopless and coloopless matroid M such that* $\operatorname{rank}(M) \leq |F|$*. Then,* $\operatorname{rank}(F\backslash\{x\}) = \operatorname{rank}(F)$ *for any $x \in F$.*

Proof. We first observe that $\operatorname{rank}(F) < \operatorname{rank}(M)$. To see why, suppose that $\operatorname{rank}(M) \leq \operatorname{rank}(F)$. Since F is proper, there exists some $x \in E\backslash F$. By F being a flat, we get $\operatorname{rank}(M) \leq \operatorname{rank}(F) < \operatorname{rank}(F \cup \{x\})$, contradicting the definition of the rank of a matroid. Thus, $\operatorname{rank}(F) < \operatorname{rank}(M) \leq |F|$, and so F is dependent. Now let $x \in F$. By properties of the rank function, we have $\operatorname{rank}(F) - 1 \leq \operatorname{rank}(F\backslash\{x\}) \leq \operatorname{rank}(F)$ so we either have $\operatorname{rank}(F\backslash\{x\}) = \operatorname{rank}(F) - 1$ or $\operatorname{rank}(F\backslash\{x\}) = \operatorname{rank}(F)$. Towards a contradiction, assume $\operatorname{rank}(F\backslash\{x\}) = \operatorname{rank}(F) - 1$ is true. We take two cases, and show that both lead to a contradiction. Assume first that $F\backslash\{x\}$ is independent. Then $\operatorname{rank}(F) - 1 = \operatorname{rank}(F\backslash\{x\}) = |F\backslash\{x\}| = |F| - 1$, implying that $\operatorname{rank}(F) = |F|$, a contradiction. Now assume that $F\backslash\{x\}$ is dependent. We show that x is a coloop by proving that for any independent subset

I, $I \cup \{x\}$ is independent. For any independent subset I in $F \backslash \{x\}$, $I \cup \{x\}$ is independent since $\text{rank}(F \backslash \{x\}) < \text{rank}(F)$. Otherwise, for any independent subset I not contained in $F \backslash \{x\}$, then $I \cup \{x\}$ is independent since I contains at least one element not in the closure of F, and hence $\text{rank}(I \cup \{x\}) > \text{rank}(I)$. Thus, we have a contradiction since we assumed that M is coloopless. Therefore, it must be the case that $\text{rank}(F \backslash \{x\}) = \text{rank}(F)$ as desired. □

Lemma 5.3. *Let M be a non-uniform matroid. Then there exists a nonempty proper flat F of M such that* $\text{rank}(M) \leq |F|$.

Proof. Since M is not uniform, then not every flat is independent and hence there exists some dependent flat F. The flat F is contained in some hyperplane H that is also dependent as it contains a dependent set. Thus, $|H| > \text{rank}(H) = \text{rank}(M) - 1$ so $|H| \geq \text{rank}(M)$, as desired. □

Proposition 5.4. *Suppose that M is a loopless and coloopless matroid (with a ground set E) such that every bichain is a biflag. Then, M is uniform.*

Proof. We prove the contrapositive. Suppose that M is a non-uniform matroid with size n and rank r. Then, by Lemma 5.3, there is a nonempty proper flat F of M such that $\text{rank}(M) \leq |F|$. We claim that $E \backslash F$ is a flat of $M^\perp$, and prove it by showing the addition of a new element to $E \backslash F$ strictly increases its rank in $M^\perp$. For ease of notation, we use A^c to denote the complement of any set A in the set E, and r, r^* to denote the rank functions of $M, M^\perp$, respectively. Let $x \in (F^c)^c = F$. From matroid duality, we have

$$\begin{aligned} r^*(F^c \cup \{x\}) &= r((F^c \cup \{x\})^c) + |F^c \cup \{x\}| - r(E) \\ &= r(F \cap \{x\}^c) + |F^c| + 1 - r(E) \\ &= r(F \backslash \{x\}) + |F^c| + 1 - r(E). \end{aligned}$$

By Lemma 5.2, we have $r(F) - 1 < r(F \backslash \{x\})$, implying

$$\begin{aligned} r(F \backslash \{x\}) + |F^c| + 1 - r(E) &> r(F) - 1 + |F^c| + 1 - r(E) \\ &= r((F^c)^c) + |F^c| - r(E) \\ &= r^*(F^c). \end{aligned}$$

This proves that F^c is a flat of $M^\perp$, which is nonempty and proper since F is nonempty and proper. Now consider the bichain $\mathcal{F}|\mathcal{G} = F|E \leq E|F^c$. We have that

$$\bigcap_{F|G \in \mathcal{F}|\mathcal{G}} F \cap G = (F \cap E) \cup (E \cup F^c) = F \cup F^c = E,$$

so $\mathcal{F}|\mathcal{G}$ is not a biflag, as desired. □

6 Further Research Questions

6.1 Palindromicity, Real-Rootedness, γ-Positivity and Unimodality

When M is uniform, we found an explicit formula for the Hilbert-Poincaré series $H_{A_{M,M^\perp}}(t)$, and proved that it is palindromic, real-rooted, γ-positive and unimodal. It would be interesting to investigate if $H_{A_{M,M^\perp}}(t)$ is real-rooted or unimodal for any other classes of matroids or for arbitrary loopless and coloopless matroids. Furthermore, by Proposition 6.1 below, the Hilbert-Poincaré series $H_{A_{M,M^\perp}}(t)$ is palindromic for any arbitrary loopless and coloopless matriod M, so it would be an interesting question to investigate if $H_{A_{M,M^\perp}}(t)$ is γ-positive for other classes of matroids or for arbitrary loopless and coloopless matroids.

Proposition 6.1. *The Hilbert-Poincaré series $H_{A_{M,M^\perp}}(t)$ is palindromic for any arbitrary matroid M*

Proof. By [AHK18], the conormal fan $A(\Sigma_{M,M^\perp})$ is Lefschetz. This implies that it is also hard Lefschetz, and so $H_{A_{M,M^\perp}}(t)$ is palindromic. For the definitions of Lefschetz and hard Lefschetz, we refer the reader to [ADH23]. □

Based on the values of $H_{A_{M,M^\perp}}(t)$ we compute in the table below, our guess is that $H_{A_{M,M^\perp}}$ is always real-rooted.

Table 2: The values $H_{A_{M,M^\perp}}(t)$ for some loopless and coloopless matroids M of size 6 and 7. See Appendix for the definitions of the M_i's.

M	$H_{A_{M,M^\perp}}(t)$	Real-rooted?
M_1	$1+32t^1+258t^2+32t^3+t^4$	✓
M_2	$1+29t^1+200t^2+29t^3+t^4$	✓
M_3	$1+26t^1+150t^2+26t^3+t^4$	✓
M_4	$1+26t^1+150t^2+26t^3+t^4$	✓
M_5	$1+23t^1+108t^2+23t^3+t^4$	✓
M_6	$1+20t^1+74t^2+20t^3+t^4$	✓
M_7	$1+75t^1+1143t^2+1143t^3++75t^4+t^5$	✓
M_8	$1+71t^1+986t^2+986t^3++71t^4+t^5$	✓

6.2 Relationship between $A_{M,M^\perp}$ and $A_M \otimes_{\mathbb{Z}} A_{M^\perp}$

It would also be interesting to study the relationship between $A_{M,M^\perp}$ and $A_M \otimes_{\mathbb{Z}} A_{M^\perp}$ in the non-uniform case. As we proved in Proposition 4.5, the two graded rings are isomorphic if M is uniform. In the more general case, when M is an arbitrary loopless and coloopless matroid, we have morphisms of fans $\pi : \Sigma_{M,M^\perp} \to \Sigma_M$ and $\overline{\pi} : \Sigma_{M,M^\perp} \to \Sigma_{M^\perp}$ [ADH23]. Consequently, we have a morphism of fans $\sigma : \Sigma_{M,M} \to \Sigma_M \times \Sigma_{M^\perp}$ giving us a pull-back map $\sigma^* : A_M \otimes_{\mathbb{Z}} A_{M^\perp} \to A_{M,M^\perp}$ since $A_M \otimes_{\mathbb{Z}} A_{M^\perp} \cong A(\Sigma_M \times \Sigma_{M^\perp})$. Proposition 3.20 of [ADH23] provides us with the following simple description of σ^*: for flags $\mathcal{F}, \mathcal{G}$ of nonempty proper flats of of $M, M^\perp$, respectively,

$$\sigma^*(x_{\mathcal{F}} \otimes_{\mathbb{Z}} 1) = \sum_{\mathcal{A}} x_{\mathcal{F}|\mathcal{A}}, \quad \sigma^*(1 \otimes_{\mathbb{Z}} x_{\mathcal{G}}) = \sum_{\mathcal{B}} x_{\mathcal{B}|\mathcal{G}}$$

where the sum in the left equality is over all decreasing sequences $\mathcal{A}$ such that $\mathcal{F}|\mathcal{A}$ is a biflag of M, and the sum in the right equality is over all increasing sequences $\mathcal{B}$ such that $\mathcal{B}|\mathcal{G}$ is a biflag of M. It would be of interest to study the kernel and the image of this map in hope of learning more about $A_{M,M^\perp}$.

6.3 Non-Dominance of Uniform Matroids

In [Fer+24], the following proposition is proven regrading the coefficients of the Hilbert series of the Chow ring of an arbitrary matroid:

Proposition 6.2. *Let M be a matroid of rank r and a ground set of size n. Then*

$$H_{A_M}(t) \preceq H_{A_{U_n^r}}(t)$$

where $\preceq$ denotes that the former polynomial is less than the latter polynomial coefficient-wise.

It is, then, natural to ask whether something similar holds for $H_{A_{M,M^\perp}}(t)$ where M is arbitrary. Explicitly, we are asking if for any matroid M of size n and rank r, we have

$$H_{A_{M,M^\perp}}(t) \preceq H_{A_{U_n^r,U_n^{n-r}}}(t). \tag{2}$$

We provide a counterexample. Let M be the matroid with ground set $E = \{1,2,3,4\}$ and basis $B = \{\{1,2\},\{2,3\},\{1,3\},\{2,4\},\{1,4\}\}$. We have that $n = 4$, $r = 2$, and one can explicitly compute that

$$H_{A_{M,M^\perp}}(t) = 1 + 3t + t^2.$$

Based on Corollary 4.7, we have that

$$H_{A_{U_n^r,U_n^{n-r}}}(t) = 1 + 2t + t^2.$$

It would be interesting to investigate if there special families of matroids M such that (2) holds true.

7 Appendix

Table 3: Bases of the matroids M_i in Table 2.

M	Bases(M)
M_1	$\{0,1,2\},\{0,1,3\},\{0,2,3\},\{1,2,3\},\{4,0,1\},\{4,0,2\},\{4,1,2\},\{4,0,3\},\{4,1,3\},\{4,2,3\},\{0,5,1\},$ $\{0, 5, 2\},\{5, 1, 2\}, \{0, 5, 3\}, \{5, 1, 3\}, \{5, 2, 3\}, \{4, 0, 5\}, \{4, 5, 1\}, \{4, 5, 2\}, \{4, 5, 3\}$
M_2	$\{0,1,3\},\{0,2,3\},\{1,2,3\},\{4,0,1\},\{4,0,2\},\{4,1,2\},\{4,0,3\},\{4,1,3\},\{4,2,3\},\{0,5,1\},\{0,5,2\},$ $\{5, 1, 2\}, \{0, 5, 3\}, \{5, 1, 3\}, \{5, 2, 3\}, \{4, 0, 5\}, \{4, 5, 1\}, \{4, 5, 2\}, \{4, 5, 3\}$
M_3	$\{0,1,3\},\{0,2,3\},\{1,2,3\},\{4,0,1\},\{4,0,2\},\{4,1,2\},\{4,0,3\},\{4,1,3\},\{4,2,3\},\{0,5,1\},\{0,5,2\},$ $\{5, 1, 2\}, \{0, 5, 3\}, \{5, 1, 3\}, \{5, 2, 3\}, \{4, 0, 5\}, \{4, 5, 1\}, \{4, 5, 2\}$
M_4	$\{0,1,3\},\{0,2,3\},\{1,2,3\},\{4,0,1\},\{4,0,2\},\{4,1,2\},\{4,1,3\},\{4,2,3\},\{0,5,1\},\{0,5,2\},\{5,1,2\},$ $\{0, 5, 3\}, \{5, 1, 3\}, \{5, 2, 3\}, \{4, 0, 5\}, \{4, 5, 1\}, \{4, 5, 2\}, \{4, 5, 3\}$
M_5	$\{0,1,3\},\{0,2,3\},\{1,2,3\},\{4,0,1\},\{4,0,2\},\{4,1,2\},\{4,1,3\},\{4,2,3\},\{0,5,1\},\{0,5,2\},\{5,1,2\},$ $\{0, 5, 3\}, \{5, 2, 3\}, \{4, 0, 5\}, \{4, 5, 1\}, \{4, 5, 2\}, \{4, 5, 3\}$
M_6	$\{0,1,3\},\{0,2,3\},\{1,2,3\},\{4,0,1\},\{4,0,2\},\{4,1,2\},\{4,1,3\},\{4,2,3\},\{0,5,1\},\{0,5,2\},\{5,1,2\},$ $\{0, 5, 3\}, \{5, 2, 3\}, \{4, 0, 5\}, \{4, 5, 1\}, \{4, 5, 3\}$
M_7	$\{0,1,2\},\{0,1,3\},\{0,2,3\},\{1,2,3\},\{4,0,1\},\{4,0,2\},\{4,1,2\},\{4,0,3\},\{4,1,3\},\{4,2,3\},\{0,5,1\},$ $\{0, 5, 2\}, \{5, 1, 2\}, \{0, 5, 3\}, \{5, 1, 3\}, \{5, 2, 3\}, \{4, 0, 5\}, \{4, 5, 1\}, \{4, 5, 2\}, \{4, 5, 3\}, \{0, 1, 6\},$ $\{0, 6, 2\}, \{1, 6, 2\}, \{0, 6, 3\}, \{1, 6, 3\}, \{6, 2, 3\}, \{4, 0, 6\}, \{4, 1, 6\}, \{4, 6, 2\}, \{4, 6, 3\}, \{0, 5, 6\},$ $\{5, 1, 6\}, \{5, 6, 2\}, \{5, 6, 3\}, \{4, 5, 6\}$
M_8	$\{0,1,3\},\{0,2,3\},\{1,2,3\},\{4,0,1\},\{4,0,2\},\{4,1,2\},\{4,0,3\},\{4,1,3\},\{4,2,3\},\{0,5,1\},\{0,5,2\},$ $\{5, 1, 2\}, \{0, 5, 3\}, \{5, 1, 3\}, \{5, 2, 3\}, \{4, 0, 5\}, \{4, 5, 1\}, \{4, 5, 2\}, \{0, 1, 6\}, \{0, 6, 2\}, \{1, 6, 2\},$ $\{0, 6, 3\}, \{1, 6, 3\}, \{6, 2, 3\}, \{4, 0, 6\}, \{4, 1, 6\}, \{4, 6, 2\}, \{4, 6, 3\}, \{0, 5, 6\}, \{5, 1, 6\}, \{5, 6, 2\},$ $\{5, 6, 3\}, \{4, 5, 6\}$

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