# The Infinite U(1) Symmetry in a Dilute Bose Gas

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## Abstract

Drawing on recent interest in the quasilinear approximation for classical fluids, we present two established models for the weakly-interacting Bose gas which possess an analogous infinite U(1) symmetry in Fourier space. While the first model, due to Boguliubov, is exactly solvable, an exact solution to the second is unknown. We clarify the conserved quantities associated with the U(1) symmetry via Noether's theorem. Finally, we explore approaches to diagonalize the second model Hamiltonian and consider connections to quantum integrability.

# Contents

# 1 Introduction

A century ago, the publication of S.N. Bose's 1924 article 'Planck's Law and the Light Quantum Hypothesis' followed by Einstein's papers on monatomic ideal gases proposed to the scientific community a new set of statistics for indistinguishable particles. From the unique statistical properties of the ideal Bose gas followed the prediction of a mysterious phase of matter: the Bose-Einstein condensate (BEC). Unlike fermions, bosonic wave functions do not obey the Pauli exclusion principle. As a result, weakly-interacting gases of very large numbers of bosons, when cooled to sufficiently low temperatures, will collapse to a common ground state. Only thirty years ago was a condensate achieved in the laboratory through a combination of laser cooling, magnetic trapping, and evaporative cooling, which famously led to the 2001 Nobel Prize in Physics awarded to Cornell, Wieman, and Ketterle [1]. The experimental breakthroughs of the past thirty years have led to renewed theoretical interest in ultracold atoms and the experimental realization of longstanding theoretical predictions.

In 1947, N. Boguliubov set out to understand the properties of superfluid Helium-4 in the article 'On The Theory of Superfluidity' [3]. However, his now ubiquitous result for the gapless linear excitation spectrum of BEC has since been observed in dilute ultracold gases [11]. In 1959, M. Girardeau and R. Arnowitt introduced a Boguliubov-type 'pair-approximation' to the fully-interacting model which conserves total particle number, and applied the variational method to calculate the approximate ground state and excitation spectrum of the gas [9]. In 1963, two papers by Lieb and Liniger derived an exact solution to the fully-interacting model of a one-dimensional Bose gas with periodic boundary conditions using the Bethe Ansatz, establishing an early example of a quantum integrable model [15]. In particular, they found that the Boguliubov method captures a portion of the exact spectrum [14]. The one-dimensional gas is not only of interest due to its tractable mathematical structure. Nearly one-dimensional gases may be experimentally realized via 'cigar-shaped traps," and rigorous results have obtained the spectrum of the Lieb-Liniger model in the limit of strong confinement along two directions [24].

At a far-removed length-scale of physics, mathematicians, physicists, and engineers continue to grapple with the perplexing properties of turbulent fluids. Several recent works have explored the quasilinear (QL) approximation for inhomogenous and anisotropic turbulent flows [17]. In the quasilinear approximation, we consider the interaction of a 'mean-flow' (obtained by a spatial or time average) with deviations from the mean. Crucially, QL dynamics exclude 'triadic interactions' in which waves outside the mean-flow interact with each other.

In Fourier space, this approximation decouples higher wave-modes from one another. Instead, they interact indirectly through the mean-flow. This procedure results in an 'infinite U(1) symmetry' by which the Fourier components of higher-wave modes may be *independently* rotated by an arbitrary phase factor $e^{i\theta}$. In the case of a fully linear approximation, this symmetry is associated with a family of conserved pseudomomenta, and the dynamics admit both Lan-

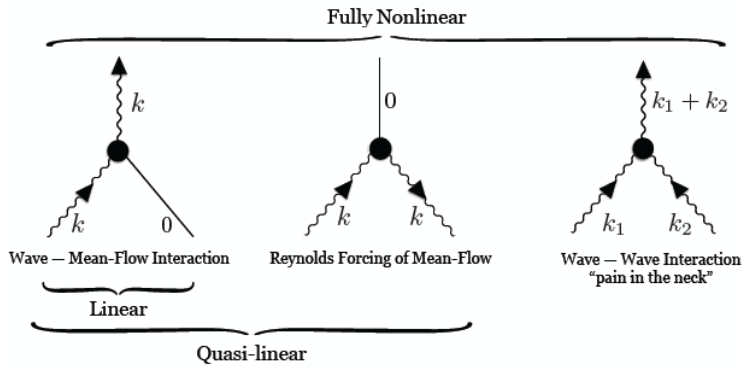



Figure 1: Triadic interactions between Fourier modes in the linear, quasilinear, and nonlinear fluid models. Reprinted from [17] with author's permission.

grangian and Hamiltonian formulations. In contrast, the quasilinear equations lack a Lagrangian or Hamiltonian formulation, so the existence of a family of conserved integrals of motion remains an open question [17]. Promisingly, a conserved quantity and a Hamiltonian formulation has been established for the 2-D vorticity equation of an inviscid, incompressible fluid on a periodic rectangle when only a single Fourier mode is retained [26].

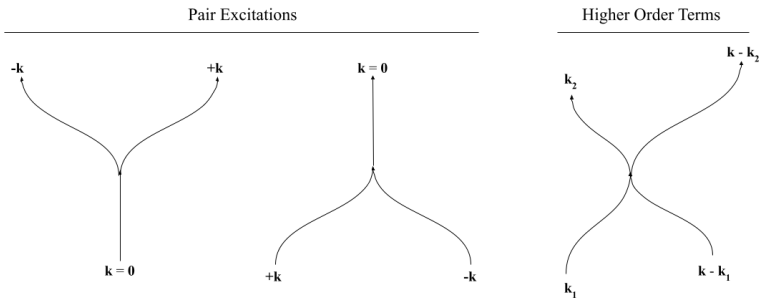



Figure 2: Boguliubov [3] and Girardeau and Arnowitt [9] both consider pair excitations, i.e. terms which are at most quadratic in the condensate operators. The fully-interacting theory includes collisions between excited states alone.

This work is motivated by an analogy between the quasilinear problem and the dilute Bose gas. Namely, the 'pair Hamiltonian' due to Girardeau and Arnowitt (GA) describes a QL-type approximation in which excited states interact in pairs with the condensate, but triadic interaction between excited states are neglected. The Boguliubov method adopts a similar scheme, but

treats the condensate as a static mean-field with which excited states of the system interact, akin to the linear approximation.

Moreover, both the Boguliubov Hamiltonian and the GA Hamiltonian exhibit an infinite U(1) symmetry in Fourier space. Whereas the former model may be exactly diagonalized by a Boguliubov transformation, an exact solution to the latter is unknown. In this work, we utilize the Hamiltonian and Lagrangian formalisms to explore this quantum analog to the QL approximation. In particular, we discuss the conserved quantities associated with the infinite U(1) symmetry, explore approaches to exactly diagonalize the GA Hamiltonian, and discuss connections between the QL approximation and the coherent dynamics of BEC.

# 2 Background

## 2.1 The Second Quantization Formalism

In this section, I offer a brief review of the mathematical setting for any suitable description of large systems of indistinguishable particles.[1]

As every physics undergraduate learns, the quantum mechanics of a single particle takes place in a Hilbert space, such as the space of complex-valued $L^2$ functions on $\mathbb{R}^3$. The quantum state of a system containing a few distinguishable particles then lives in in the tensor product space $\mathcal{H}^{\otimes N} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$, where $N$ denotes the number of particles.

However, when treating large systems which are better described in terms of fields than particles or when dealing with thermodynamic ensembles in which particles are freely exchanged with the environment, it is extremely useful to develop a formalism which is flexible with respect to the total number of particles. Moreover to capture the unique statistical properties of bosons, we must only consider states which are symmetric under particle exchange.

Let $\{|\psi_k\rangle\}_{k\in K}$ denote an orthonormal basis for $\mathcal{H}$. (The notation is chosen to suggest the momentum basis, although $K$ can be any indexing set in general.)

Then we define the following operator:

**Definition 2.1** (Permutation Operator)**.** The permutation operator $\hat{P}_{ij}^N$ is the self-adjoint linear operator which acts on a basis element of $\mathcal{H}^{\otimes N}$ as follows:

$$\hat{P}_{ij}^N \, |\psi_{k_1}\rangle \otimes \ldots |\psi_{k_i}\rangle \ldots |\psi_{k_j}\rangle \cdots \otimes |\psi_{k_N}\rangle \equiv |\psi_{k_1}\rangle \otimes \ldots |\psi_{k_j}\rangle \ldots |\psi_{k_i}\rangle \cdots \otimes |\psi_{k_N}\rangle$$

i.e. by permuting the states of particle $i$ and particle $j$.

It follows from this definition that $\hat{P}_{ij}^2 = I$ (the identity). Moreover, $\hat{P}_{ij}$ may only have eigenvalues of $\pm 1$. As we are interested in systems of bosons and not fermions, we will only consider bosonic states which yield an eigenvalue of 1 under a permutation. Although, per our definition $\hat{P}_{ij}$ only effects a transposition,

[1] For a gentle introduction see [23]. For more detailed and/or rigorous discussions see [12], [19], or [13].

this suffices to describe a representation of the entire symmetric group $\mathcal{S}_N$ on $\mathcal{H}^{\otimes N}$ [2].

To restrict the larger space occupied by distinguishable particles, we define a new operator which yields bosonic states.

**Definition 2.2** (Symmetrizer Operator)**.** The symmetrizer on $\mathcal{H}^{\otimes N}$ is defined by its action on a basis:

$$\hat{S}^N |\psi_{k_1}\rangle \otimes \cdots \otimes |\psi_{k_N}\rangle \equiv \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathcal{S}_N} |\psi_{k_{\sigma(1)}}\rangle \otimes \cdots \otimes |\psi_{k_{\sigma(N)}}\rangle$$

In the above expression, $\frac{1}{\sqrt{N!}}$ is a normalization factor, since we sum over $N!$ permutations. The state of a system of $N$ bosons then occupies the symmetric subspace:

$$\mathcal{H}^N_{sym} \equiv \hat{S}_N(\mathcal{H}^{\otimes N})$$

Moreover the image of the tensor product basis reduces to an orthogonal basis of the symmetric subspace. In particular, if the single-particle basis is countable, then choosing $K = \mathbb{N}$, we can represent a basis element as follows:

$$|n_1, n_2, n_3, ...\rangle_{sym} = \hat{S}^N \left( \bigotimes_{k \in \mathbb{N}} |\psi_k\rangle^{\otimes n_k} \right)$$

where

$$\sum_{k \in \mathcal{N}} n_k = N$$

.

The above expression means that a symmetric basis state is completely specified by the number of particles in each single particle state. However, the states as presented above are not orthonormal, as we have not taken into account redundancies due to multiple particles occupying the same state.

A direct calculation yields:

$${}_{sym}\langle n_1, n_2, n_3, \ldots | n_1, n_2, n_3, \ldots \rangle_{sym} = (n_1!)(n_2!)\ldots$$

Thus, we define the bosonic basis states as follows:

$$|n_1, n_2, n_3, ...\rangle \equiv \frac{1}{\sqrt{\prod_{k \in K}(n_k)!}} |n_1, n_2, n_3, ...\rangle_{sym} \tag{2.1}$$

From this definition, orthonormality and completeness relations for this basis of a *fixed* number of bosons may be verified. We are now equipped to address states with variable particle number.

**Definition 2.3** (Fock Space)**.** Let $\mathcal{H}$ denote the Hilbert space of a single particle, which admits the basis $|\psi_k\rangle_{k \in K}$. Then the bosonic Fock space is:

$$\mathcal{F} \equiv \bigoplus_{N=1}^{\infty} \mathcal{H}^N_{sym} = \mathbb{C} \oplus \mathcal{H}_{sym} \oplus \mathcal{H}^2_{sym} + \ldots$$

Again, as above, we express a general basis state in $\mathcal{F}$ as:

$$|n_1, n_2, n_3, ...\rangle$$

However, the total number of particles may be any whole number. In particular, there is a 'vacuum' state $|0\rangle \equiv |n_1 = 0, n_2 = 0, \ldots\rangle$ corresponding to no particles occupying the system. We remark that $|0\rangle$ is *not* the zero vector and is in fact a normalized basis element of $\mathcal{F}$. We define the inner product on $\mathcal{F}$ such that basis states of different total particle number are orthogonal.

Having established the states with which we will be working, we proceed to discuss the operators acting on those states. Many-body Hamiltonians often feature one-body operators (such as kinetic energy and momentum) and two-body operators (Coulomb interactions, spin-spin interactions, etc.). Moreover, for each single-particle state $|\psi_k\rangle$, we may define non-hermitian operators which 'create' or 'annihilate' particles in that state.

**Definition 2.4** (Creation/Annihilation Operators)**.** The annihilation operator acts a basis state as follows:

$$\hat{a}_k |n_1, \ldots, n_k, \ldots\rangle = \sqrt{n_k} |n_1, \ldots, n_k - 1, \ldots\rangle$$

Where $\hat{a}_k |0\rangle = 0$. The creation operator is the adjoint of $\hat{a}_k$:

$$\hat{a}_k^\dagger |n_1, \ldots, n_k, \ldots\rangle = \sqrt{n_k + 1} |n_1, \ldots, n_k + 1, \ldots\rangle$$

It follows that $\hat{a}_k^\dagger |0\rangle = |0, \ldots, n_k = 1, \ldots\rangle$. Together these form a self-adjoint 'number operator' of which every basis element is an eigenstate:

$$\hat{n}_k \equiv \hat{a}_k^\dagger \hat{a}_k$$

This operator simply counts the number of particles in the $|\psi_k\rangle$ state. It also allows us to form the 'total number operator':

$$\hat{N} = \sum_{k \in K} \hat{n}_k$$

Finally, it remains to state the key properties of the creation/annihilation operators which form the bosonic algebra.

**Proposition 2.1.** The set of operators $\{\hat{a}_k, \hat{a}_k^\dagger\}_{k \in K}$ satisfies the following relations:

1. $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$
2. $[\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0$
3. $[\hat{a}_j, \hat{a}_k] = 0$

*Proof.* These identities follow directly from Definition 2.4, so I will only demonstrate the first claim. If $j \neq k$, then:

$$\hat{a}_j \hat{a}_k^\dagger |n_1, \ldots\rangle = \sqrt{n_j}\sqrt{n_k + 1} |n_1, \ldots, n_j - 1, n_k + 1\rangle = \hat{a}_k^\dagger \hat{a}_j |n_1, \ldots\rangle$$

Thus the commutator gives the zero vector. Whereas, if $j = k$, then:

$$\hat{a}_k \hat{a}_k^\dagger |n_1, \ldots\rangle = \sqrt{n_k + 1}\sqrt{n_k + 1} |n_1, \ldots, n_k, \ldots\rangle = (\hat{n}_k + 1) |n_1, \ldots\rangle$$

Since $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$, we find the commutator gives the identity on a basis vector. Since this result holds for every basis vector, it uniquely determines the commutator over $\mathcal{F}$. □

The heuristic definitions in this section enable us to powerfully formulate the evolution of many-body systems, which still obeys the Schrodinger equation, where $\hat{H}$ becomes a many-body operator acting on an evolving Fock state. In the Heisenberg representation, we can study the dynamical evolution of operators instead:

$$i\hbar \frac{d\hat{A}(t)}{dt} = [\hat{A}, \hat{H}]$$

For a fixed operator $\hat{A}$ acting on a state which evolves under the Schrodinger flow:

$$\hat{A}(t) = \hat{U}^\dagger(t) \hat{A} \hat{U}(t)$$

where $\hat{U}$ is the unitary time evolution operator generated by $\hat{H}$.

This motivates the following definition.

**Definition 2.5** (Conserved Quantity)**.** A conserved quantity in quantum mechanics is an operator $\hat{A}$ such that:

$$[\hat{A}, \hat{H}] = 0$$

In the next section we present the most important theorem about conserved quantities in classical mechanics.

## 2.2 Noether's Theorem: A Quick Review

In 1918, Emmy Noether published a paper on *Invariante Variationsprobleme* [20], a profound result in variational calculus and the theory of differential invariants with deep implications for physicists. At the heart of Noether's theorem is a connection between symmetries of a variational problem and conserved quantities of the equations of motion. Below, we present a special case of the theorem, which we will make use of later on.[2]

[2] See [22] for an rich discussion of the history, subtleties, and consequences of Noether's two theorems.

**Theorem 2.2** (Noether's First Theorem - Abridged)**.** Consider a functional, $S$, which depends on a single independent variable $t \in \mathbb{R}$, and the functions $q = (q_1(t), \ldots, q_k(t))$ and their first derivatives $\dot{q}$.

$$S[q] \equiv \int_\Omega \mathcal{L}(t, q, \dot{q})\, dt$$

Let $\mathcal{G}$ be a one-parameter group which acts $(t, q)$ such that $S$ is invariant under the action of $G$ for an arbitrary region $\Omega$. This means, $\forall \rho \in \mathcal{G}$, $S[\rho q] = S[q]$. Consider an infinitesimal variation $\rho_\epsilon(t, q) = (t + \epsilon T, q + \epsilon Q) + o(\epsilon^2)$, where $\epsilon = 0$ corresponds to the identity in $\mathcal{G}$. Then for all $q$ which solve the variational problem $\frac{\delta S}{\delta q} = 0$, the following identity holds:

$$\frac{d}{dt}\big\{(\mathcal{L} - \sum_{i=1}^{k} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i)T + \sum_{i=1}^{k} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} Q\big\} = 0 \tag{2.2}$$

*Proof.* Denote $(t', q') = (t + \epsilon T, q + \epsilon Q)$. In what follows, we only retain terms to first order in $\epsilon$.

First consider:

$$S[\rho_\epsilon q] = \int_{\Omega'} \mathcal{L}(t', q'(t'), \dot{q}'(t'))\, dt$$

By partial integration we can change the bounds to:

$$S[\rho_\epsilon q] = \int_\Omega \mathcal{L}(t, q'(t), \dot{q}'(t)) + \frac{d}{dt}(\mathcal{L}\epsilon T)\, dt$$

Then, invariance of the integral stipulates:

$$0 = S[\rho_\epsilon q] - S[q] = \int_\Omega \mathcal{L}(t, q'(t), \dot{q}'(t)) - \mathcal{L}(t, q(t), \dot{q}(t)) + \frac{d}{dt}(\mathcal{L}\epsilon T)$$

Note that while $q'(t) = q(t) + \epsilon Q$, we have $q'(t) = q'(t' - \epsilon\delta t) = q(t) + \epsilon Q - \epsilon T\dot{q}$. We denote this variation at a constant time by $\bar{\delta} q = \epsilon Q - \epsilon T \dot{q}$. Then

$$\begin{aligned}\mathcal{L}(t, q'(t), \dot{q}'(t)) - \mathcal{L}(t, q(t), \dot{q}(t)) &= \sum_{i=1}^{k} \frac{\partial \mathcal{L}}{\partial q_i}\bar{\delta} q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\bar{\delta}\dot{q}_i \\ &= \sum_{i=1}^{k} \big(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\big)\bar{\delta} q_i + \frac{d}{dt}\big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\bar{\delta} q_i\big)\end{aligned}$$

For classical trajectories $q$, the Euler-Lagrange equations imply that the first two terms above vanish. Thus, we find:

$$0 = \int_\Omega \frac{d}{dt}\big\{(\mathcal{L}\epsilon T + \sum_{i=1}^{k} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\bar{\delta} q_i\big\}\, dt$$

Since this relation holds over any region $\Omega$, the integrand must be zero almost everywhere. Thus we find:

$$\frac{d}{dt}\{(\mathcal{L} - \sum_{i=1}^{k} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i)T + \sum_{i=1}^{k} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} Q\} = 0$$

□

Note that the terms proportional to $T$ are a Legendre transformation of the Lagrangian, giving the Hamiltonian. For time-independent group actions, this term will be zero.

Although the theorem, as stated above, is a one-way implication, Noether proved a two-way relation between variational symmetries and divergence relations [20]. However, Harvey Brown notes that:

> Not all differential equations in physics have a variational (Lagrangian) formulation; in such cases the application of Noether's theorem is ruled out. Examples are the Navier-Stokes equations (in most cases) and Fourier's heat equation. Furthermore, the very question whether a given dynamical theory has a Lagrangian formulation can depend on how the relevant equations of motion, or field equations, are formulated. [4]

In this way, studying a quantum mechanical system which is automatically endowed with a Hamiltonian offers distinct advantages in analyzing the connection between symmetries and conserved quantities. In the next section we demonstrate how to obtain a Lagrangian from a many-body Hamiltonian.

## 2.3 Coherent States and the Path Integral

A useful tool in dealing with bosonic systems are 'coherent states," which can be defined as eigenvectors of the annihilation operator. We will use these states to derive a Lagrangian for the many-body system via a path-integral approach. This approach also permits a semiclassical analysis of the system. For the upshot, skip to equations 2.5-2.6.

In the discussion below, I closely follow the derivation in [19].

**Proposition 2.3.** For every vector of complex numbers $\{\phi_k\}_{k\in K}$, there exists a Fock state $|\{\phi_k\}\rangle$ such that $\forall j \in K$, $\hat{a}_j |\{\phi_k\}\rangle = \phi_j |\{\phi_k\}\rangle$.

To understand this proposition, consider a single momentum state:

$$|n_k = N\rangle = |n_0 = 0, \ldots, n_k = N, n_k + 1 = 0, \ldots\rangle = \frac{(\hat{a}_k^\dagger)^N}{N!} |0\rangle$$

Applying $\hat{a}_k$ to any *basis state* yields an orthogonal state, since the particle number is different. Thus, in order to satisfy the above property coherent states must be composed of states with arbitrarily large particle number.

We construct a coherent state as follows:

*Proof.* Let $|\phi_k\rangle = \sum_{n=1}^{\infty} \frac{(\phi_k \hat{a}_k^\dagger)^n}{n!} |0\rangle$. Then $\hat{a}_k |\phi_k\rangle = \phi_k |\phi_k\rangle$. Thus:

$$|\{\phi_k\}\rangle = \prod_{k \in K} \sum_{n_k=1}^{\infty} \frac{(\phi_k \hat{a}_k^\dagger)^n}{n!} |0\rangle$$

□

We represent such a state in the form:

$$|\{\phi_k\}\rangle = e^{\sum_{k \in K} \phi_k \hat{a}_k^\dagger} |0\rangle$$

A number of useful properties immediately avail themselves from the above. For instance, $\langle\{\phi_k\}| \hat{a}_k^\dagger = \langle\{\phi_k\}| \phi_k^*$ and $\hat{a}_k^\dagger |\{\phi_k\}\rangle = \frac{\partial}{\partial \phi_k} |\{\phi_k\}\rangle$.

While coherent states are non-orthogonal and in fact form an *overcomplete* basis for $\mathcal{F}$, they allow us to represent many-body operators in terms of complex numbers and analytic functions.

In order to discuss a path integral over coherent states, we must define the completeness relation, stated without proof or derivation below:

$$\int \prod_{k \in K} \frac{d\phi_k^* d\phi_k}{2\pi i} e^{-\sum_K |\phi_k|^2} |\{\phi_k\}\rangle \langle\{\phi_k\}| = \mathbb{1}$$

Where the right-hand side is the identity operator. Note, at each $k$, we integrate over the complex plane spanned by $\phi_k$ and $\phi_k^*$. In doing so, we treat complex conjugates as independent integration variables. In real and complex parts, the integration measure is an iterated sequence of double integrals over $x$ and $y$ [27]:

$$\phi_k = \frac{1}{\sqrt{2}}(x_k + i y_k) \tag{2.3}$$

$$\phi_k^* = \frac{1}{\sqrt{2}}(x_k - i y_k) \tag{2.4}$$

We proceed to derive an explicit form for the propagator in the coherent state basis.

$$U(\phi^*(t_1), \phi(t_0)) = \langle \phi(t_1) | e^{-\frac{i}{\hbar} \hat{H}(t_1 - t)} | \phi(t_0) \rangle$$

We then subdivide the time interval $(t, t_1)$ into $M$ intervals and define $\Delta t = \frac{t_1 - t}{M}$ in order to re-write evolution operator as a product:

$$\begin{aligned} U(\phi^*(t_1), \phi(t_0)) &= \langle \phi(t_1) | \, e^{-\frac{i}{\hbar} \hat{H} \Delta t} \prod_{j=1}^{M-1} (\mathbb{1} e^{-\frac{i}{\hbar} \hat{H} \Delta t}) \, |\phi(t_0)\rangle \\ &= \int \mathcal{D}_{M-1}[\phi^*, \phi] \prod_{j=1}^{M-1} e^{-\sum_K |\phi_{j,k}|^2} \prod_{j=1}^{M} \langle \phi_j | e^{-\frac{i}{\hbar} \hat{H} \Delta t} | \phi_{j-1} \rangle \end{aligned}$$

Where in the above we define:

$$\mathcal{D}_{M-1}[\phi^*, \phi] = \prod_{j=1}^{M-1} \prod_{k \in K} \frac{d\phi^*_{j,k} d\phi_{j,k}}{2\pi i}$$

Thus we have inserted $M-1$ dummy integration variables between $\phi_0 = \phi(t_1)$ and $\phi_M = \phi(t_0)$.

Neglecting terms of order $(\Delta t)^2$, we can write the unitary evolution operator as a normally-ordered operator in $\hat{a}_k^{(\dagger)}$. We then replace creation and annihilation operators by their eigenvalues $\hat{a}_k^{(\dagger)} \to \phi_k^{(*)}$:

$$\langle \phi_j | e^{-\frac{i}{\hbar}\hat{H}(\hat{a}_k^\dagger, \hat{a}_k)\Delta t} | \phi_{j-1} \rangle = \langle \phi_j | e^{-\frac{i}{\hbar}H(\phi_j^*, \phi_{j-1})\Delta t} e^{\sum_{k \in K} \phi^*_{j,k}\phi_{j-1,k}} | \phi_{j-1} \rangle$$

The path integral is then obtained by taking a 'limit' as $M \to \infty$. [3] In this limit, we may view the exponentiated terms as a Riemann sum over the parametrized path over coherent states $\phi(t)$ where $t$ ranges continuously over $(t_0, t_1)$.

Letting $\frac{\phi_{j,k} - \phi_{j-1,k}}{\Delta t} \to \frac{d\phi_k}{dt}$ some simplification yields the following expression:

$$U(\phi^*(t_1), \phi(t_0)) = \int \mathcal{D}[\phi^*, \phi] e^{\frac{i}{\hbar}S[\phi^*, \phi] + |\phi(t_1)|^2} \tag{2.5}$$

Where in the exponent we have the 'classical action' of the system:

$$S[\phi^*, \phi] = \int_{t_0}^{t_1} \sum_{k \in K} i\hbar \phi_k^* \partial_t \phi_k - H(\phi_k^*, \phi_k)\, dt \tag{2.6}$$

The integrand in the above is the Lagrangian of our system in the coherent-state basis. The reader may recognize 2.5-2.6 as a kind of phase-space path integral. For each $k \in K$, we have a generalized coordinate $\phi_k$ and its corresponding conjugate variable $\phi_k^*$. The Euler-Lagrange equations then yield semi-classical trajectories of the system.

Setting $\hbar \equiv 1$, we find these equations are equivalent to Hamilton's equations in complex variables [25]:

$$i\partial_t \phi_k = \frac{\partial H}{\partial \phi_k^*}$$

For a system, such as ours, with a countable, yet infinite, basis $K$, this yields an infinite set of coupled complex-valued ODEs in the real variable $t$. As we will see, even for a finite system, these equations may be impossible to solve. However, this correspondence allows complicated relations in the bosonic algebra to be easily verified in terms of commuting complex variables. Moreover, the existence of a Lagrangian enables us to directly compute the conserved charges associated with the generator of a continuous symmetry.

[3] As Sigal and Gustafson note, this limit is not well-defined. However, the Wick-rotated counterpart of this integral (which transforms the Hamiltonian into a partition function), may be rigorously defined using the Wiener integral. [13]

## 2.4 The Dilute Boson Gas

The starting point for our discussion of the dilute boson gas must be the second quantized Hamiltonian.

$$\hat{H} = \int \hat{\psi}^\dagger(\vec{x}) \frac{\hbar^2}{2m} \nabla^2 \hat{\psi}(\vec{x}) \, d\vec{x} + \frac{g}{2} \int \hat{\psi}^\dagger(\vec{x}) \hat{\psi}^\dagger(\vec{y}) \delta(\vec{x} - \vec{y}) \hat{\psi}(\vec{y}) \hat{\psi}(\vec{x}) \, d\vec{x} d\vec{y} \quad (2.7)$$

In the previous exposition of the second quantization formalism, we largely avoided reference to the 'local field operators." These are 'operator-valued distributions' [13], i.e. generalized functions of the spatial variables which yield an operator on the symmetric Fock space $\mathcal{F}$ and satisfy their own bosonic commutation relations:

$$[\hat{\psi}(\vec{x}), \hat{\psi}^\dagger(y)] = \delta(\vec{x} - \vec{y}) \quad (2.8)$$

Physically, we interpret $\hat{\psi}^\dagger \hat{\psi}$ as an operator which yields the density of particles in a region of the configuration space. Thus,

$$\hat{N} = \int_{\mathbb{R}^3} \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x}) \, d\vec{x} \quad (2.9)$$

In 2.7, the first term corresponds to the kinetic energy, while the second is a 'self-interaction' term, which describes short-range repulsive $\delta$-potential scattering. $g$ then determines the (im)penetrability of particles with respect to one another. As $g \to \infty$, we discover the 'hard-core' limit, whereas the limit $g \to 0$ eliminates any inter-particle interactions, so particles are free to move through one another.

### 2.4.1 Deriving the Boguliubov Hamiltonian

Traditionally, the Boguliubov theory of the Bose gas is used to derive the 'low-energy' spectrum in which the kinetic energy of individual particles is small compared to the coupling strength. In this limit, we choose $g = 4\pi a^2$, where $a$ is the S-wave scattering length determined by the Born approximation [11].

Suppose we confine our gas of particle to a box of volume $\Omega$ and impose periodic boundary conditions on $\hat{\psi}$. One can then verify directly the spectrum of the $g = 0$ limit to be $\frac{\hbar^2 k^2}{2m}$, for $k \in \frac{\pi}{\Omega}\mathbb{Z}^3$.

We are now ready to present a version of Boguliubov's original calculation. First, he inserts the Fourier expansion into 2.7:

$$\hat{\psi}(\vec{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{k}} \hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$$

Note that $\hat{a}_k = \int \hat{\psi}(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \, d\vec{x}$ [4] is the annihilation operator of momentum states which we discussed earlier. By a straightforward calculation, one may

[4] Constant factors are included in the integration measure.

verify that the relation 2.8 implies 2.1 and vice-versa. Applying the boson commutation relations yields:

$$\hat{H} = \sum_{\vec{k}} \epsilon_{\vec{k}} \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \frac{u}{2} \sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4} \hat{a}^\dagger_{\vec{k}_1} \hat{a}^\dagger_{\vec{k}_2} \hat{a}_{\vec{k}_3} \hat{a}_{\vec{k}_4} \delta_{\vec{k}_1+\vec{k}_2,\vec{k}_3+\vec{k}_4} \tag{2.10}$$

Where we define $\epsilon_{\vec{k}} \equiv \frac{\hbar|\vec{k}|^2}{2m}$ and $u \equiv \frac{g}{\Omega}$.

This equation defines the fully-interacting model of the gas. Using the translation invariance of 2.7, we can boost into the rest frame of the condensate, so that the mode $\vec{k} = \vec{0}$ corresponds to the condensate.

At extremely cold temperatures, a macroscopic number of particles occupy the $\vec{k} = \vec{0}$ state, while far fewer occupy excited states. Thus $\langle \hat{n}_0 \rangle \gg \langle \hat{N} - \hat{n}_0 \rangle$.

Let us then expand $\hat{H}$ to separate out terms which include the condensate operators:

$$\hat{H} = \frac{u}{2}(\hat{a}^\dagger_0)^2(\hat{a}_0)^2 + \sum_{\vec{k}\neq 0}(\epsilon_k + \hat{a}^\dagger_0\hat{a}_0)\hat{a}^\dagger_k\hat{a}_k \tag{2.11}$$

$$+ \frac{u}{2}\sum_{\vec{k}\neq 0}(a^\dagger_k a^\dagger_{-k}(\hat{a}_0)^2 + (\hat{a}^\dagger_0)^2\hat{a}_k\hat{a}_{-k}) \tag{2.12}$$

$$+ u \sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3\neq 0} \big(\hat{a}^\dagger_{\vec{k}_1}\hat{a}^\dagger_{\vec{k}_2}\hat{a}_{\vec{k}_3}\hat{a}_0 + \hat{a}^\dagger_{\vec{k}_3}\hat{a}^\dagger_0\hat{a}_{\vec{k}_1}\hat{a}_{\vec{k}_2}\big)\delta_{\vec{k}_1+\vec{k}_2,\vec{k}_3} \tag{2.13}$$

$$+ \frac{u}{2} \sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4\neq 0} \hat{a}^\dagger_{\vec{k}_1}\hat{a}^\dagger_{\vec{k}_2}\hat{a}_{\vec{k}_3}\hat{a}_{\vec{k}_4}\delta_{\vec{k}_1+\vec{k}_2,\vec{k}_3+\vec{k}_4} \tag{2.14}$$

In the first line of 2.14, we re-write

$$(\hat{a}^\dagger_0)^2(\hat{a})^2 = \hat{a}^\dagger_0(\hat{a}_0\hat{a}^\dagger_0 - 1)\hat{a}_0 = \hat{n}_0(\hat{n}_0 - 1)$$

Boguliubov's radical approach was to make the replacement $\hat{a}^{(\dagger)}_0 \to \sqrt{N}$, where $N$ is simply a constant (c-number). Formally, this substitution does not seem to make sense. As we have seen, the creation and annihilation operators applied alter the number of particles in a basis state, yielding an orthogonal state. The loose justification is that the ground state of the non-interacting gas is an eigenvector of $\hat{a}_0$. Thus the substitution assumes a coherent state-like ground state.

Proceeding with the substitution, we retain only those terms of order $N$ or $N^2$, dropping terms of order $\sqrt{N}$ or lower:

$$\hat{H}_B = \frac{u}{2}N(N-1) + \sum_{k\neq 0}(\epsilon_k + uN)\hat{a}^\dagger_k\hat{a}_k + \frac{uN}{2}(\hat{a}^\dagger_k\hat{a}^\dagger_{-k} + \hat{a}_k\hat{a}_{-k}) \tag{2.15}$$

This is the Boguliubov Hamiltonian, which is not only far simpler, but can be exactly diagonalized by means of a Boguliubov transformation.

### 2.4.2 Diagonalizing $\hat{H}_B$

We begin with a few simple identities which justify the definition that follows.

**Proposition 2.4.** Let $\hat{b}_k = c_k \hat{a}_k - s_k \hat{a}^\dagger_{-k}$, for $c_k, s_k \in \mathbb{R}$. Then $[\hat{b}_k, \hat{b}_j] = 0$. Moreover, if $c_k^2 - s_k^2 = 1$, then $[\hat{b}_k, \hat{b}_j^\dagger] = \delta_{k,j}$

The transformation described above is a special case of the Boguliubov transformation. In our case, we can use this transformation to define new bosonic quasiparticle operators which provide a diagonal basis for $H_B$.

$$\hat{b}_k = \cosh(\gamma_k)\hat{a}_k - \sinh(\gamma_k)\hat{a}^\dagger_{-k} \tag{2.16}$$

$$\hat{a}_k = \cosh(\gamma_k)\hat{b}_k + \sinh(\gamma_k)\hat{b}^\dagger_{-k} \tag{2.17}$$

We may then directly compute terms of $\hat{H}_B$:

$$\begin{aligned}
\hat{a}_k^\dagger \hat{a}_k =& \cosh^2(\gamma_k)\hat{b}_k^\dagger \hat{b}_k + \sinh^2(\gamma_k)\hat{b}_{-k}\hat{b}_{-k}^\dagger \\
&+ \frac{1}{2}\sinh(2\gamma_k)(\hat{b}_k\hat{b}_{-k} + \hat{b}_k^\dagger \hat{b}_{-k}^\dagger) \\
\hat{a}_k^\dagger \hat{a}_k^\dagger + \hat{a}_k\hat{a}_k =& \cosh(2\gamma_k)(\hat{b}_k\hat{b}_{-k} + \hat{b}_k^\dagger \hat{b}_{-k}^\dagger) \\
&+ \sinh(2\gamma_k)(1 + \hat{b}_k^\dagger \hat{b}_k + \hat{b}_{-k}^\dagger \hat{b}_{-k})
\end{aligned}$$

To simplify notation, let $\xi_k \equiv \epsilon_k + uN$ and $\eta_k \equiv uN$. Then the Boguliubov Hamiltonian (with a little rearranging) is re-written in terms of the phonon operators as:

$$\begin{aligned}
H_B =& \frac{u}{2}N(N-1) \\
&+ \sum_{\vec{k}\neq 0} \frac{1}{2}(\xi \sinh(2\gamma_k) + \eta\cosh(2\gamma_k))(\hat{b}_k\hat{b}_{-k} + \hat{b}_k^\dagger\hat{b}_{-k}^\dagger) \\
&+ \sum_{\vec{k}\neq 0} \xi\sinh^2(2\theta_k) + \frac{\eta}{2}\sinh(2\theta_k) \\
&+ \sum_{\vec{k}\neq 0} \hat{b}_k^\dagger\hat{b}_k(\xi\cosh(2\gamma_k) + \eta\sinh(2\gamma_k))
\end{aligned}$$

So far, we have not fixed the angle $\gamma_k$. To get rid of off-diagonal terms, we require $\xi\sinh(2\gamma_k) + \eta\cosh(2\gamma_k) = 0$. Thus we define:

$$\tanh(2\gamma_k) = -\frac{\eta}{\xi} = -\frac{uN}{\epsilon_k + uN} \tag{2.18}$$

Where $\gamma_k = \gamma_{-k}$. It follows that:

$$\xi\cosh(2\gamma_k) + \eta\sinh(2\gamma_k) = \sqrt{\xi^2 - \eta^2} = \sqrt{\epsilon_k^2 + 2uN\epsilon_k}$$

Applying this to the Hamiltonian we find:

$$\hat{H} = \frac{u}{2}N(N-1) + E' + \sum_{k\neq 0} E(k)\hat{b}_k^\dagger \hat{b}_k \tag{2.19}$$

In which the ground state correction is:[5]

$$E' = \sum_{\vec{k}\neq 0} \xi \sinh^2(\theta_k) + \frac{\eta}{2}\sinh(2\theta_k) = \frac{1}{2}\sum_{\vec{k}\neq 0} E(k) - (\epsilon_k + uN) + \frac{(uN)^2}{2\epsilon_k} \tag{2.20}$$

and the Boguliubov dispersion law is given by:

$$E(k) = \sqrt{\epsilon_k^2 + 2uN\epsilon_k} \tag{2.21}$$

In the limit $\epsilon_k \ll uN$, we discover the famous linear dispersion regime:

$$E(k) \approx \sqrt{2uN\epsilon_k} = \hbar k\sqrt{\frac{uN}{m}}$$

# 3 Infinite U(1) Symmetry of the Bose Gas

Despite the remarkable success of Boguliubov's method, the c-number substitution $\hat{N} \mapsto N$ alters an important symmetry of the problem: particle number conservation. While $\hat{N} = \sum_{\vec{k}} \hat{n}_k$ commutes with the fully-interacting Hamiltonian in 2.10, it does *not* commute with $H_B$.

This is because, under the evolution of $H_B$, the condensate does not fluctuate, i.e. $[\hat{n}_0, H_B] = 0$. Instead, the condensate acts as a static mean field with which pairs of excited momentum states interact.

In this section we discuss a particle number-conserving approximation to the fully-interacting Hamiltonian. We then apply Noether's theorem to understand its symmetries in relation to $\hat{H}_B$.

## 3.1 Two Number-Conserving Approaches to Boguliubov Theory

In [7], C.W. Gardiner revises the traditional approach to a weakly interacting Bose gas by introducing new operators $\alpha_k$, $\alpha_k^\dagger$ which commute with the total number operator. This yields an approximate Hamiltonian for the dilute Bose gas which is very similar in form to the Boguliubov Hamiltonian. We define:

$$\alpha_k \equiv \frac{a_0^\dagger a_k}{\sqrt{\hat{N}}} \tag{3.1}$$

[5] The last term in 2.20 is a higher-order correction which renormalizes the scattering length and effective potential [21]. This resolves the ultraviolet divergence in the expression we would derive from the previous discussion alone.

Where $\sqrt{\hat{N}}$ is the unique positive square root of the positive self-adjoint operator $\hat{N}$.

These operators evidently preserve particle number:

$$[\alpha_k, \hat{N}] = \frac{1}{\sqrt{\hat{N}}}[a_0^\dagger a_k, a_0^\dagger a_0 + a_k^\dagger a_k] = \frac{1}{\sqrt{\hat{N}}}(a_0^\dagger a_k - a_0^\dagger a_k) = 0$$

Moreover, in the regime in which the Boguliubov replacement is valid,

$$\left\langle \alpha_k^\dagger \alpha_k \right\rangle \approx \left\langle \hat{a}_k^\dagger \hat{a}_k \right\rangle$$

The Gardiner Hamiltonian is then written:

$$H_G = \frac{u}{2}\hat{N}(\hat{N}-1) + \sum_{k\neq 0}(\epsilon_k + u\hat{N})\alpha_k^\dagger \alpha_k + \frac{u}{2}\hat{N}(\alpha_k \alpha_{-k} + \alpha_k^\dagger \alpha_{-k}^\dagger) \tag{3.2}$$

This Hamiltonian now commutes with $\hat{N}$, and may, in principle, be diagonalized within a subspace of fixed $N$.

However, the $\alpha_k$ only behave approximately like bosonic operators:

$$[\alpha_k, \alpha_{k'}^\dagger] = \frac{1}{\hat{N}}[a_0^\dagger a_0 a_k a_{k'}^\dagger - (1 + a_0^\dagger a_0) a_k^\dagger a_{k'}] = \frac{1}{\hat{N}}(n_0 \delta_{k,k'} - a_{k'}^\dagger a_k) \tag{3.3a}$$

$$\approx \delta_{k,k'} - \frac{1}{\hat{N}} a_{k'}^\dagger a_k \approx \delta_{k,k'} \tag{3.3b}$$

Where 3.3b holds for highly condensed states where $\frac{n_0}{N} \approx 1$ and $\frac{n_k}{N} \ll 1$.

Although, we could replicate the discussion in Section 2.4 with the Gardiner Hamiltonian, the approximate nature of the above commutation relations makes an exact analysis difficult, yielding corrections in inverse powers of $\sqrt{\hat{N}}$. A subsequent article by S.A. Gardiner and S.A. Morgan revises this approach by expanding in inverse powers of the condensate fraction, rather than the total number of particles [8]. This self-consistent method allows for a better accounting of condensate depletion.

However, for the rest of this work, we will use an older model Hamiltonian introduced by Girardeau and Arnowitt (GA) in [9] for three reasons: (1) its relative simplicity, (2) the use of exact bosonic operators, and (3) it matches the spectrum of the fully-interacting Hamiltonian exactly in the limit $u \to 0$, even for states with significant condensate depletion. A detailed comparison of the GA model with C.W. Gardiner's model can be found in [10].

The GA Hamiltonian is simply given by the first two lines of 2.14:

$$H_{GA} = \frac{u}{2}N(N-1) + \sum_{\vec{k}\neq 0}\left\{(\epsilon_k + u\hat{n}_0)\hat{n}_k + \frac{u}{2}\left(\hat{a}_k^\dagger \hat{a}_{-k}^\dagger (\hat{a}_0)^2 + \hat{a}_k \hat{a}_{-k}(\hat{a}_0^\dagger)^2\right)\right\} \tag{3.4}$$

$H_{GA}$ manifestly conserves particle number, which justifies the replacement $\hat{N} \to N$. However, the condensate is no longer fixed. Thus the quadratic

Hamiltonian $H_B$ is replaced with a simplified *quartic* Hamiltonian in which excited modes interact indirectly through a fluctuating condensate, analogous to the QL approximation.

As we will see in the next section $H_B$, $H_G$, and $H_{GA}$ possess a common set of symmetries. Whereas the fixed condensate of the former yields an exactly diagonalizable model, we will investigate whether the latter Hamiltonian permits a similar calculation.

## 3.2 The U(1) Symmetry

The symmetry group with which we are concerned is the circle group:

$$U(1) = \{e^{i\theta} | \theta \in \mathbb{R}\}.$$

In quantum mechanics, this generally corresponds to a phase rotation of the wave function, or one of its components. Many quantum mechanical systems exhibit number-phase uncertainty relations and the U(1) symmetry is often associated with a number conservation laws.

As many authors note, the fully-interacting model of the Bose gas is invariant under a global U(1) transformation. In terms of the local field operators, we express this as follows:

$$\hat{\psi} \mapsto e^{i\theta}\hat{\psi} \qquad \hat{\psi}^\dagger \mapsto e^{-i\theta}\hat{\psi}^\dagger$$

Equivalently, $\forall k \in K$ we may send $\hat{a}_k \mapsto e^{i\theta}\hat{a}_k$. Note that 2.10 is invariant under this transformation.

By a first-order Taylor expansion, we see the infinitesimal generator of this phase rotation is $\delta\hat{\psi} = i\hat{\psi}\delta\theta$. Applying Noether's theorem, we find that the associated conserved quantity is $\hat{N} = \int \hat{\psi}^\dagger\hat{\psi}\, d\vec{x}$.

Further, as many authors note, Bose-Einstein condensation spontaneously breaks this global U(1) symmetry. Hence the ground state may be specified by a coherent state in which the particle number is not fixed, but has a coherent phase [11].

Let us now consider the free, non-interacting Bose gas, whose Hamiltonian is given by:

$$\hat{H}_{free} = \sum_{k\neq 0} \epsilon_k \hat{a}_k^\dagger \hat{a}_k$$

This Hamiltonian possesses a stronger symmetry than the previous: it is invariant under an *infinite* U(1) symmetry, $U(1)^{\otimes K}$, which applies a phase rotation independently at each wave mode. $\forall k \in K$, we have:

$$\hat{a}_k \mapsto e^{i\theta_k}\hat{a}_k \qquad \hat{a}_k^\dagger \mapsto e^{-i\theta_k}\hat{a}_k^\dagger$$

$\hat{H}_{free}$ has an infinite set of conserved quantities corresponding to the number of particles occupying each momentum state:

$$[\hat{n}_k, \hat{H}_{free}] = 0$$

Shortly, we will illustrate the connection using Noether's theorem. However, we note that in this case, we have a diagonal Hamiltonian, and each symmetry-conserved quantity pair corresponds to an invariant subspace of the Hamiltonian.

Turning to the Boguliubov theory, it is clear the replacement $\hat{a}_0^{(\dagger)} \mapsto \sqrt{N}$ violates the global U(1) symmetry. Yet, a *reduced* infinite U(1) symmetry emerges. In this case, each wave-mode cannot be phase-shifted independently. Rather, the nature of the 'pair excitations' in the Boguliubov theory permits a coupled phase shift of the following form:

$$\hat{a}_k \mapsto e^{i\theta_k}\hat{a}_k \qquad \hat{a}^\dagger_{-k} \mapsto e^{i\theta_k}\hat{a}^\dagger_{-k}$$

We may think of $\hat{H}_B$ as invariant under the action of $U(1)^{\otimes(K/\sim)}$ where $\sim$ is the equivalence relation on $K$ defined by $\vec{k} \sim -\vec{k}$.

The number-conserving Hamiltonians are similarly invariant under the reduced infinite U(1) symmetry. However, neither $H_G$ nor $H_{GA}$ are invariant under a phase-shift of $\hat{a}_0$, so we exclude the $\vec{k} = 0$ state and replace it with invariance under a global $U(1)$ transformation.

Finally, we arrive at the central motivating question of this work. Under a particle number-conserving Boguliubov approximation, given by $H_{GA}$, we see both the manifestation of a global U(1) symmetry and the reduced infinite U(1) symmetry of $H_B$. What are the conserved quantities associated with these symmetries, and when do they enable us to exactly diagonalize the Hamiltonian?

The magic of the Boguliubov transformation is that by diagonalizing $H_B$, we lift the *reduced* U(1) symmetry to the infinite U(1) symmetry of $H_{free}$, whereby the system exhibits an infinite set of independent conserved quantities at each wave mode, in this case corresponding to the number of quasiparticle excitations, rather than particle excitations.

In the section that follows, we apply Noether's theorem to illustrate (1) the direct connection between the U(1) symmetry and its associated conserved quantities for $H_B$, $H_{free}$, and $H_{GA}$, which explains (2) why we cannot know *a priori* that $H_{GA}$ may be directly diagonalized as a result of its symmetries.

## 3.3 Conserved Charges of the U(1) Symmetry

In sections 2.2-2.3, we developed the basic theory to permit a straightforward application of Noether's theorem.

Let us begin with $\hat{H}_{free}$. This Hamiltonian is associated with the following Lagrangian in the coherent state basis:

$$\mathcal{L}(\phi_k^*, \phi_k) = \sum_{k\in K} \left(i\phi_k^*\partial_t\phi_k - \epsilon_k\phi_k^*\phi_k\right)$$

If we independently vary each phase $\theta_k$, we obtain the following (infinite) set of conserved charges:

$$Q_k = \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_j)} \frac{\partial \phi_j}{\partial \theta_k} = \sum_j (i\phi_k^*)(i\phi_k \delta_{j,k}) = -|\phi_k|^2$$

Of course, $|\phi_k|^2 = \langle \phi | \hat{n}_k | \phi \rangle$. Accordingly, $\hat{n}_k$ is the conserved quantity associated with the U(1) invariance of $H_{free}$.

Now, we progress to the reduced U(1) symmetry of $H_B$. There, our Lagrangian is:

$$\mathcal{L}(\phi^*(t), \phi(t)) = \sum_k i\phi_k^* \partial_t \phi_k - (\epsilon_k + uN)|\phi_k|^2 - \frac{uN}{2}(\phi_k^* \phi_{-k}^* + \phi_k \phi_{-k}) \quad (3.5)$$

Thus the same calculation yields:

$$Q_k = \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_j)} \frac{\partial \phi_j}{\partial \theta_k} = \phi_{-k}^* \phi_{-k} - \phi_k^* \phi_k = -(|\phi_k|^2 - |\phi_{-k}|^2)$$

We conclude $\hat{n}_k - \hat{n}_{-k}$ is the associated conserved quantity. This conservation law is a result of the pair structure of excitations in the Boguliubov theory. Yet, as we have noted, the Boguliubov transformation enables a remarkable decoupling of the phases of opposite momentum states. Whereas prior to transformation, the reduced U(1) symmetry requires simultaneous rotation of $\hat{a}_k$, $\hat{a}_{-k}^\dagger$, $\hat{a}_{-k}$, and $\hat{a}_k^\dagger$, the diagonalized form decouples the first two from the last two. From equations 2.17, we see that the reduced infinite $U(1)$ symmetry generates the following transformation:

$$\hat{b}_k \mapsto e^{i\theta_k} \hat{a}_k \qquad \hat{b}_{-k} \mapsto e^{-i\theta_k} \hat{b}_{-k}$$

However,in the quasiparticle basis, we may rotate $\hat{b}_k$ and $\hat{b}_{-k}$ by independent phases, lifting the requirement $\theta_k = \theta_{-k}$.

Here is another way to see what has occurred: one may directly compute the quasiparticle number operator $\hat{n}_k^{(B)} \equiv \hat{b}_k^\dagger \hat{b}_k$ in terms of $\hat{a}_k^{(\dagger)}$. Then the following identity holds:

$$\hat{n}_k - \hat{n}_{-k} = \hat{n}_k^{(B)} - \hat{n}_{-k}^{(B)} \quad (3.6)$$

Whereas the left-hand side represents the conserved quantity of the reduced U(1) symmetry, the operators on the right-hand side are independently conserved. This is the troubling fact we are left with: the reduced U(1) symmetry only yields trivial conserved quantities, unless transformed into an appropriate basis in which a stronger U(1) invariance may be realized.

For the Girardeau-Arnowitt Hamiltonian, $\hat{n}_k - \hat{n}_{-k}$ remains the conserved quantity associated with the reduced U(1) symmetry. Whether a transformation exists which splits the symmetry in two remains an open question. The ansatz used by Girardeau and Arnowitt in [9] sheds light on the difficulty of exactly solving the model. First they define:

$$\hat{\beta}_0 = \hat{a}_0 \hat{n}_0^{-1/2} \qquad \hat{\beta}_0^\dagger = \hat{n}_0^{-1/2} \hat{a}_0^\dagger \quad (3.7)$$

These are unitary operators which raise or lower the condensate number by 1. With these operators we may define a number-conserving variant of the Boguliubov quasiparticle:

$$\hat{B}_k^\dagger = c_k \hat{a}_k^\dagger \hat{\beta}_0 - s_k \hat{b}_{-k} \hat{\beta}_0^\dagger$$

Where $c_k = \cosh\theta_k$, $s_k = \sinh\theta_k$. This transformation is a particularly elegant choice because not only do they exactly satisfy the bosonic commutation relations, but we may verify that defining $\hat{n}_k^{(\beta)} \equiv \hat{B}_k^\dagger \hat{B}_k$, the following property holds:

$$\hat{n}_k - \hat{n}_{-k} = \hat{n}_k^{(\beta)} - \hat{n}_{-k}^{(\beta)} \tag{3.8}$$

However, a tedious direct calculation shows that the $[n_k^{(\beta)}, H]$ does not straightforwardly vanish for some choice of $\theta_k$. In the coherent state basis, we can try to directly calculate $\frac{d}{dt}\langle n_k^{(\beta)}\rangle$. However, even restricting to a single excited mode, we find that the angle $\theta_k$ that causes the derivative to vanish depends on $|\phi_0|^2$ in general. Thus 3.7 fails to yield a time-independent transformation which diagonalizes the Hamiltonian. In [9], Girardeau and Arnowitt apply the variational method to determine the appropriate transformation in the form specified above, and use this ansatz to approximately diagonalize the pair Hamiltonian, finding corrections to the Boguliubov dispersion.

In the next section, I describe various attempts, both numerical and analytical to study the effects of the reduced U(1) symmetry on the $H_{GA}$ dynamics.

# 4 Results

Two distinct approaches were used to ascertain the existence of a conserved quantity corresponding to each non-zero wave mode. For the Boguliubov Hamiltonian, the size and dimension of the set $K$ is irrelevant to the construction of conserved quantities, as coupling only occurs between opposite momentum states. Although the GA Hamiltonian exhibits indirect coupling between excited states, in order to simplify the problem, we restrict $K$ to a finite set by imposing a cut-off on the maximum kinetic energy of any given particle.

As a convenient shorthand, we define the number of excited modes, up to equivalence ($\sim$) as $f \equiv \frac{1}{2}(|K| - 1)$. For instance, the system with $f = 1$ effectively occupies 1 dimension, and each particle may access momentum states 0 (condensate) or $\pm 1$ (first excited states).

For the numerical experiments in the sections that follow, I restrict to one spatial dimension, and rescale parameters such that $K \subset \mathbb{Z}$ and $\epsilon_k = |k|^2$. Thus, there are only two momentum states corresponding to the kinetic energy $\epsilon_k$, while the Hamiltonian depends on a single tunable parameter $u$.

## 4.1 Lagrangian Equations of Motion

Using the Lagrangian formulation developed in previous sections, we derive the classical equations of motion for the system in the coherent state basis.

$$\mathcal{L}_{GA} = \sum_k i\phi_k^* \partial_t \phi_k - \frac{u}{2} N(N-1) \\ - \sum_{k\in K} \left\{ (\epsilon_k + u|\phi_0|^2)|\phi_k|^2 + \frac{u}{2}\left(\phi_k^* \phi_{-k}^* (\phi_0)^2 + \phi_k \phi_{-k} (\phi_0^*)^2\right)\right\} \quad (4.1)$$

In the above expression, there is an ambiguity about whether to treat $N$ as a c-number or an operator. Although, [9] treats this term as a pure constant, below I treat $N = \sum_{k\in K} |\phi_k|^2$ as a function of the independent variables.[6]

To simplify notation, we also define the condensate and non-condensate numbers: $N_c = |\phi_0|^2$ and $N_{nc} = N - N_c$.

The Euler-Lagrange equations for the system read as follows:

$$\partial_t \phi_k^* = i\left\{ \left( u(N - \frac{1}{2} + N_c) + \epsilon_k \right)\phi_k^* + u\phi_{-k}(\phi_0^*)^2 \right\} \quad (4.2a)$$

$$\partial_t \phi_0^* = i\left\{ \left( u(N - \frac{1}{2} + N_{nc}) \right)\phi_0^* + u\sum_{j\neq 0} \phi_j^* \phi_{-j}^* \phi_0 \right\} \quad (4.2b)$$

From these equations the following identities may be derived:

$$\frac{dN}{dt} = 0 \quad (4.3)$$

$$\frac{d}{dt}|\phi_k|^2 - |\phi_{-k}|^2 = 0, \qquad \forall k \in K \quad (4.4)$$

When restricted to a finite number of excited modes, equations 4.2 form a finite-dimensional complex-valued dynamical system, in which the analogy with the classical notion of 'complete integrability' is particularly apparent.

The free Hamiltonian of the previous chapter possesses $|K| = 2f + 1$ generalized coordinates and $2f + 1$ independent conserved quantities. (Note that while $\hat{N}$ and $\hat{H}$ are both conserved, in this case they may be written as a linear combination of the conserved operators $\hat{n}_k$).

In contrast, the reduced U(1) problem appears to give us a serious undercount: $f+2$ conserved quantities. ($f$ for the conserved difference in number, and two more corresponding to $N$ and $H$.) However, when $f = 1$, $2f+1 = f+2$, so we have exactly one conserved quantity for each coordinate degree of freedom. The correspondence with complete integrability is made explicit by introducing the Poisson bracket for complex variables [25]:

$$\{A, B\}_{(\phi_k^*, \phi_k)} = i\sum_{k\in K} \frac{\partial A}{\partial \phi_k^*}\frac{\partial B}{\partial \phi_k} - \frac{\partial A}{\partial \phi_k}\frac{\partial B}{\partial \phi_k^*} \quad (4.5)$$

[6] This choice results in the $u(N - \frac{1}{2})$ terms in 4.2. While this term alters the rate of phase rotation in the complex plane, it does not affect the behavior of $|\phi|^2$ in time.

The time-dependence of any observable is then given by:

$$\dot{S} = \{S, H\}$$

For $S = \phi_k$ this equation reproduces 4.2. For the $f = 1$ case, we label the coordinates $\phi_0, \phi_+, \phi_-$, and for convenience we define $D = |\phi_+|^2 - |\phi_-|^2$. By a direct calculation, one may verify that $\{N, H\} = 0$, $\{D, H\} = 0$, and $\{D, N\} = 0$. In short, we have a phase space of 6 real dimensions and three conserved quantities $D, N, H$ in involution. If $\nabla$ denotes the 6-gradient, then when $\nabla H, \nabla N, \nabla D$ are linearly independent, the system is *completely integrable* [18]. Under these conditions, the Liouville-Arnold theorem guarantees that the motion of this system will live on the torus $T^3$ and that a set of action-angle variables (locally) exist which exactly specify the motion [18]. Whether these action-angle variables may be analytically computed remains an open question, as the construction involves a suitable choice of coordinates. (The action variables are defined as integrals over basic cycles of the level-energy torus.) It is also possible that there is a transformation which will diagonalize this system exactly.

Nonetheless, this system is easily solved by a numerical integrator for various $f$. As expected, the $f = 1$ system executes orderly, quasiperiodic motion, whereas for $f > 1$, the motion is more complex.

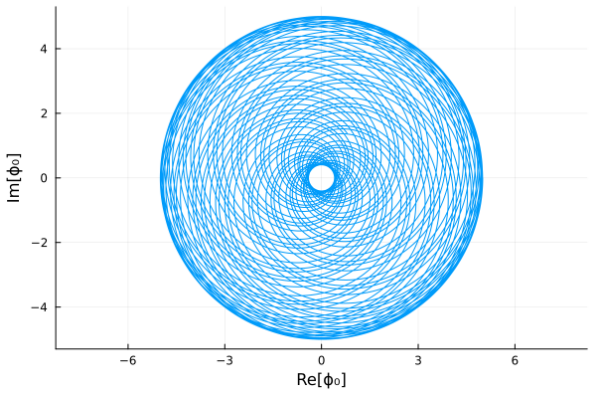
Figure 3b indicates that condensate number fluctuates periodically, and the non-condensate occupation numbers fluctuate in anti-phase with the condensate. Greater values of the coupling constant, $u$ generally correspond to an increase in the frequency of oscillations. Note also that the amplitude of these oscillations does not span the full range of motion. That is to say, the motion does not necessarily reach states in which the condensate is totally depleted or totally occupied. For initial conditions in which $|\phi_0|^2 = N$, the motion is constant, which we can see from equations 4.2 themselves.

Plotting $\phi_0$ in the complex plane (Figure 3a) demonstrates the underlying quasi-periodic motion, which traces the interior of and eventually fills an annulus. In this respect, the dynamics differ from the analogous equations of motions derived from the Boguliubov Lagrangian in 3.5. Plotting the phase space projections of those equations, we see closed periodic orbits rather than the quasi-periodic motion above.

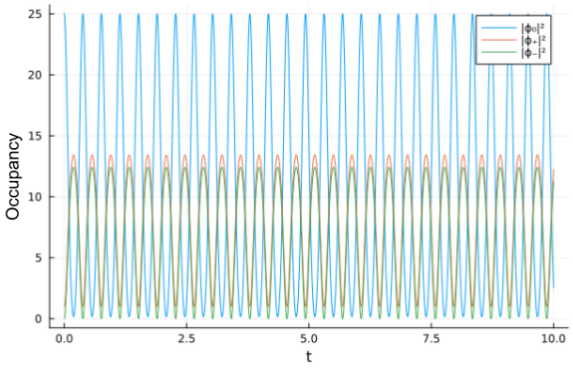
For the GA system, numerical experiments indicate that a phase-shift of the initial conditions leaves the plots of $|\phi|^2$ unchanged, while rotating the Argand diagram. As noted earlier, the term $\frac{u}{2}N(N-1)$ does not affect the plot of $|\phi|^2$, only the underlying frequency of rotation in the complex plane.

A frequency analysis of the numerical data or an exact analysis of the equations of motions given by 4.2 may also yield better insights into the characteristic frequencies of motion, and their dependence on both $u$ and the initial conditions.

In the case of $f > 1$ excited modes, the behavior appears to be more complex. One nice feature of the pair approximation, as compared to a more general type of quasilinear approximation, is that these equations do not immediately spawn higher modes. Thus the $f = 1$ system is really a system with arbitrarily large




(a) Condensate executes quasiperiodic motion in the complex plane




(b) Condensate and non-condensate occupation numbers oscillate at regular intervals

Figure 3: Regular motion of a system with $f = 1$, $N = 26$

$f$ such that the initial conditions for all higher modes are set to 0. In general, even a small perturbation[7] of the $f = 1$ system due to the presence of another mode can cause dramatic fluctuations.

However, for small values of $u$, corresponding to a small perturbation of the non-interacting limit $u \to 0$ the system does appear to produce quasiperiodic motion, with only a slight appearance of disorder.[8] We might see this behavior as a manifestation of the Kolmogorov-Arnold-Moser (KAM) theorem which describes the persistence of invariant tori under perturbations of an integrable system.

[7] We can consider perturbations such that $|\phi_k|^2 \ll |\phi_0|^2$, $|\phi_{\pm 1}|^2$, and even $|\phi_1|^2 - |\phi_{-1}|^2$.

[8] 'Small' values of the coupling constant $u$ are defined in comparison to the kinetic energy of excited particles $\epsilon_k$. In our units, this translates to $u \ll 1$, as the energy of the first excited state is 1.

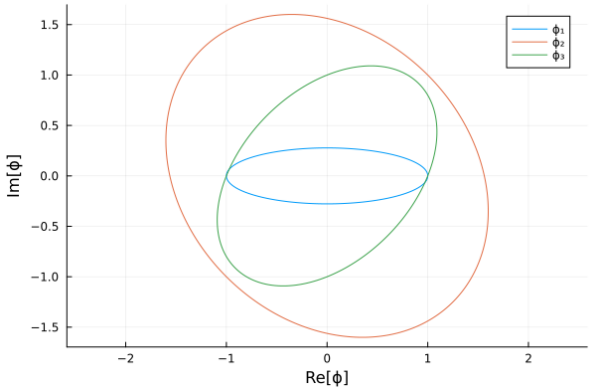



Figure 4: Closed periodic orbits of the Boguliubov system, with $f = 3$, $N = 6$.

In general, this analysis shows that even in cases where $N_{nc} \ll N_c$ initially, the condensate number can fluctuate dramatically and irregularly over time for a system containing few excited modes.

Determining the presence of quasiperiodic motion in the case of multiple interacting modes requires a more sophisticated frequency analysis. Further numerical investigations could also estimate Lyapunov exponents for trajectories in the many-mode system. The presence of chaos would provide strong evidence for the failure of complete integrability. In turn, this would suggest that the quantum problem is non-integrable, at least in its classical limit.

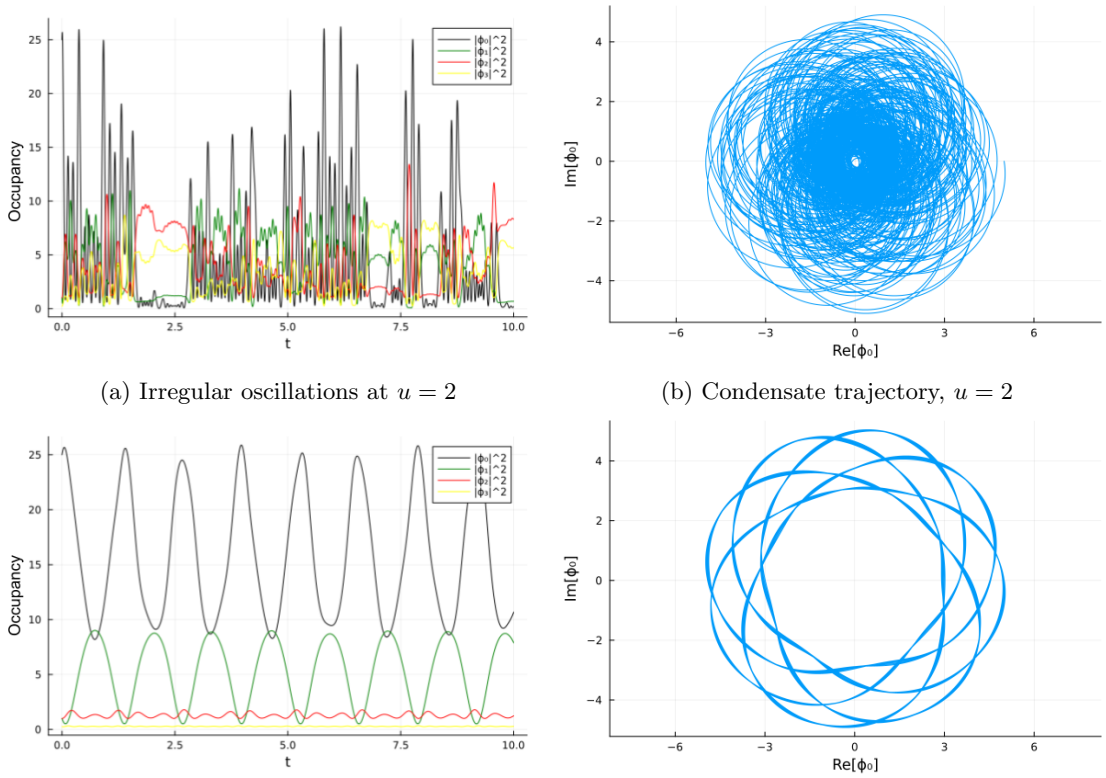
On the other hand, an analytical solution to the integrable $f = 1$ system may offer insights into hidden symmetries of the $f > 1$ systems.

## 4.2 Matrix Representations of the Hamiltonian

An alternative approach to the classical methods described above is exact diagonalization, which retains the quantum mechanical formalism while restricting to a finite-dimensional subspace of the full Fock space. In these subspaces we can directly compute matrix representations and spectra for each model Hamiltonian.

Let us begin by considering the Girardeau-Arnowitt Hamiltonian. We first restrict to a subspace of fixed total particle number $N$, and then further restrict to a finite number of modes $f$ by zeroing out any matrix elements between basis states containing particles in excited states beyond $f$. That is to say, we only consider states such that if $|k| > f$, $\hat{n}_k |\psi\rangle = 0$. This subspace is divided into invariant subspaces of the operators $\hat{n}_k - \hat{n}_{-k}$. We restrict to the subspace spanned by states which are accessible to the state $|n_0 = N\rangle$, in which the condensate is fully occupied. In other words, this is the subspace of 0 total momentum. The resulting subspace is finite-dimensional, and we can directly compute both a basis and the matrix representation of $H_{GA}$.

Below, we present a sample calculation, with $f = 1$. We denote a basis state




(a) Irregular oscillations at $u = 2$

(b) Condensate trajectory, $u = 2$

(c) Nearly periodic oscillations at $u = 0.25$

(d) Condensate trajectory, $u = 0.25$

Figure 5: Dynamics of $f = 3$ system with identical initial conditions at different coupling strengths.

as $|n_0, n_1\rangle$ where $n_1 = n_{+1} = n_{-1}$. This means the number of excited particles is $2n_1$ and $N = n_0 + 2n_1$.

Then the action of $H_{GA}$ on a generic basis element is given by:

$$\begin{aligned} H_{GA} = |n_0, n_1\rangle = \big(\frac{u}{2}N(N-1) + 2(\epsilon_1 + un_0)n_1\big)\,|n_0, n_1\rangle \\ + u(n_k+1)\sqrt{n_0(n_0-1)}\,|n_0-2, n_1+1\rangle \\ + un_k\sqrt{(n_0+1)(n_0+2)}\,|n_0+2, n_1-1\rangle \end{aligned}$$

Using this equation, we can fix $N$ and compute the matrix representation of $H$ within that subspace.

For example, if we fix $N = 2$ then there are only two basis states:

$$\mathcal{B} = \{|2,0\rangle, |0,1\rangle\}$$

$H_G$ acts on the basis vectors as follows:

$$H_{GA}\,|2,0\rangle = u\,|2,0\rangle + \sqrt{2}u\,|0,1\rangle$$
$$H_{GA}\,|0,1\rangle = \sqrt{2}u\,|2,0\rangle + (u+2\epsilon_1)\,|0,1\rangle$$

We represent $H_{GA}$ in the $\mathcal{B}$ basis as:

$$M_{N=2,f=1} = \begin{pmatrix} u & \sqrt{2}u \\ \sqrt{2}u & u+2\epsilon_1 \end{pmatrix}$$

This procedure is straightforwardly generalized to arbitrary $N$ and $f$. See Section 8 for the Python implementation.

Calculating matrix representations of $H_B$ is very similar. For fixed $N, f$, the subspaces of $H_{GA}$ and $H_B$ are isomorphic. The subtlety involved is that in the Boguliubov theory we assume $N$ is a constant which replaces the condensate occupation number. By fixing $N$, we only consider basis states with *at most* $N$ excited particles.

By this construction, the subspace dimension will be:

$$|\mathcal{B}| = \frac{(\frac{N}{2}+f)!}{\frac{N}{2}!f!}$$

This is simply the number of ways to allocate $\frac{N}{2}$ particles between $f+1$ states. Given the pair nature of excitations under $H_B$ and $H_{GA}$, this specifies all configurations of the system.

We may also compute matrix representations for the fully-interacting Hamiltonian given by 2.10. For the example above the basis for $H_{FI}$ will be the same as for $H_{GA}$. However, as we go to larger $f$, the bases grow much faster because we do not just include pair excitations.

In general, we would have:

$$|\mathcal{B}| = \frac{(N+2f)!}{N!(2f)!}$$

However, we replace the $f$ constraints that $n_k - n_{-k} = 0$, with one constraint corresponding to total momentum conservation:

$$\sum_{k=1}^{f}(n_k - n_{-k}) = 0$$

which reduces the dimension of the system. Nonetheless, the rapid growth of these matrices leads to very slow computation for large $N$, $f$, which we performed using the Second Quantization package in Sympy.

In Figure 6, we compare the eigenvalues of $H_B$, $H_{GA}$, and $H_{FI}$ at different values of $u$. A few interesting features emerge. In [9], Girardeau and Arnowitt liken their work to an 'intermediate-coupling approximation' and note that: 'The resultant low-lying spectrum lies below that of Bogoliubov by an amount

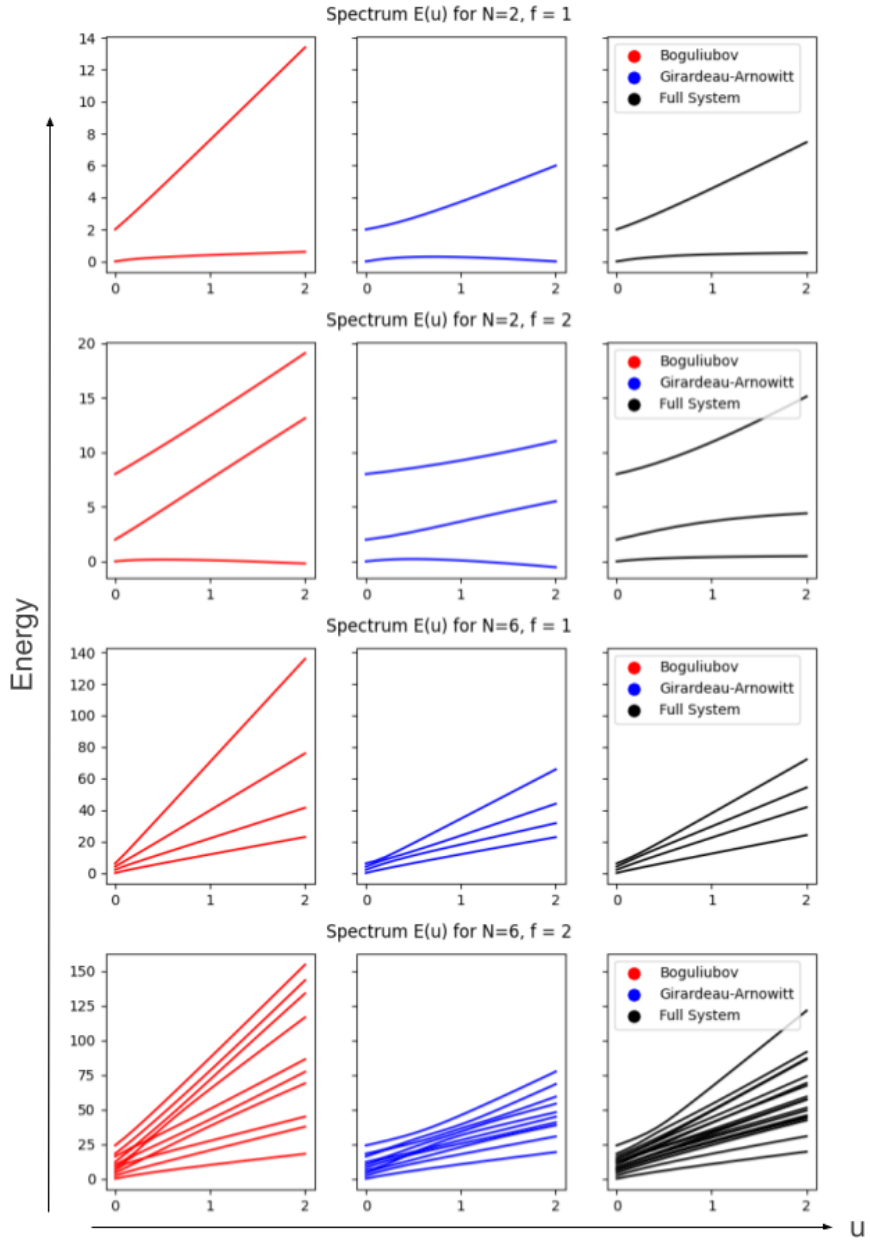



Figure 6: Spectral Comparison of $H_B$, $H_{GA}$, and $H_{FI}$ at different choices of $N$ and $f$. Energies are plotted against the coupling constant $u$.

proportional to the total number of particles.' These observations ring true to the behavior we see in the spectral comparisons, where half of the Boguliubov spectrum tends to vastly overshoot, while the Girardeau-Arnowitt spectrum effectively captures more of the level-splitting which occurs in the fully-interacting system. Nonetheless, $H_{GA}$, by only including pair interactions, ignores certain triadic and quartic interactions.

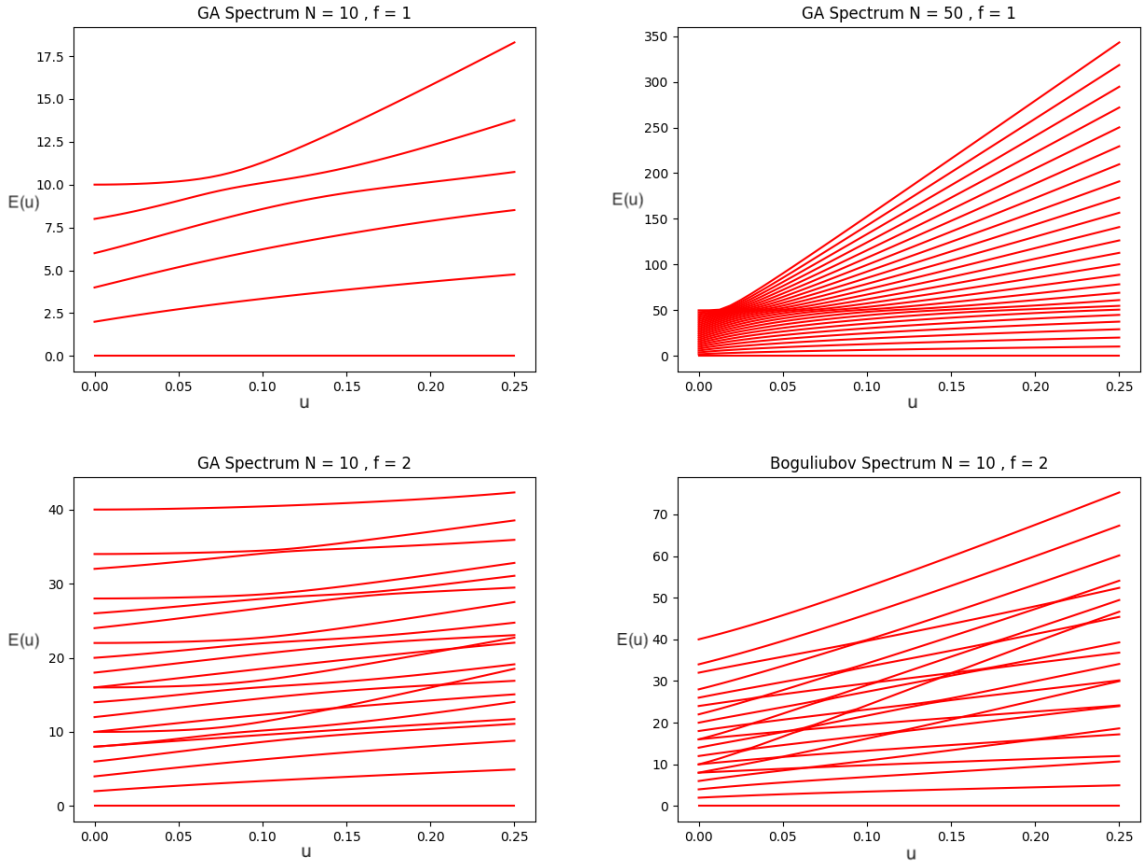
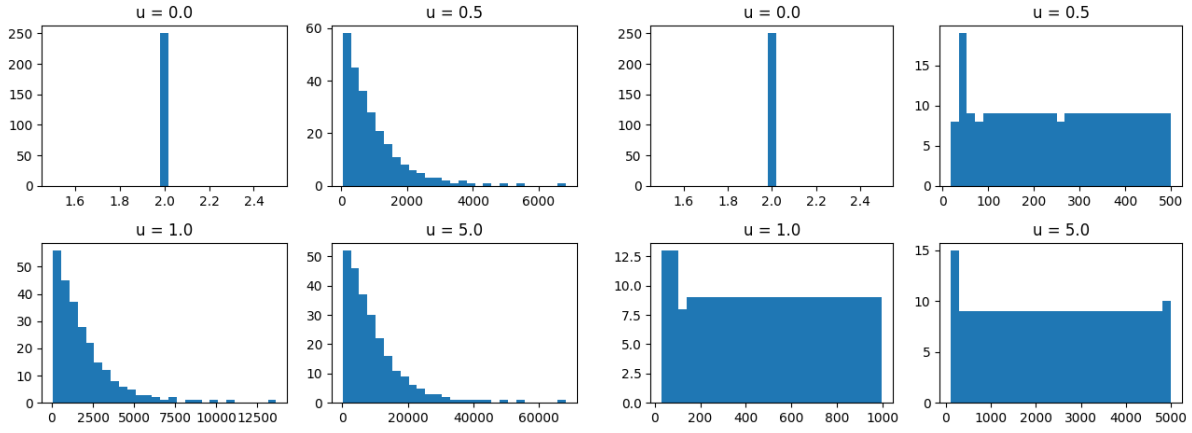



Figure 7: In (a)-(b) we see that crossings are avoided (obeying the Wigner-Von Neumann rule) for $f = 1$. In (c)-(d) we see that level crossings emerge with more excited modes, an potential indicator of independent underlying excitations.

In 7, we see level repulsion for the $f = 1$ case in both the Boguliubov and Girardeau-Arnowitt systems. However, for $f > 1$ level crossings appear to occur in both approximate Hamiltonians, while they are avoided in the fully-interacting system. In general, level crossings (i.e. degenerate eigenvalues) at certain values of the parameter $u$ are indications of underlying symmetries. We may speculate that the absence of level-crossings in the $f = 1$ case is due to the

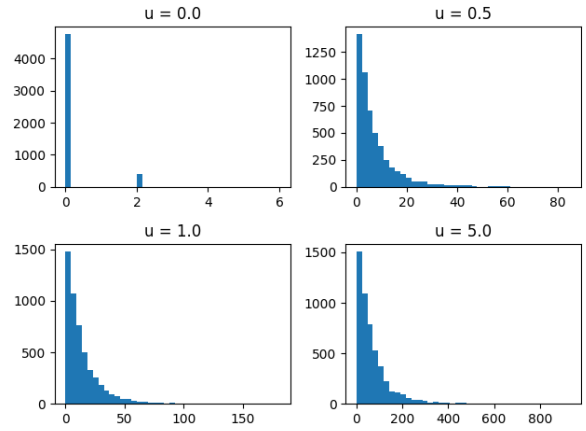
fact that we have already applied a maximal set of independent conservation laws. In contrast, the many-mode systems appear to have further symmetries that remain unaccounted for, perhaps corresponding to additional conserved quantities. Nevertheless, this intuitive argument requires further investigation to be rigorously substantiated.



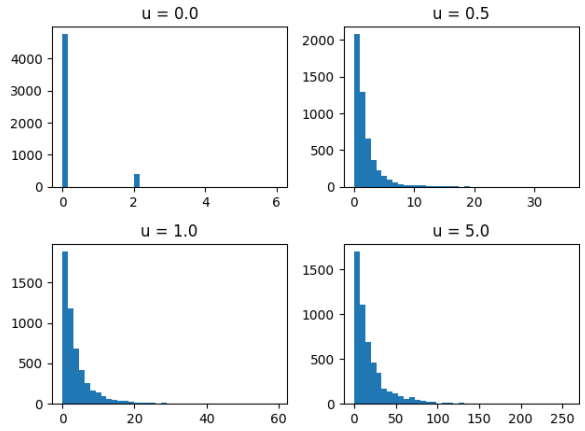

(a) $H_B$, N = 500, f = 1




(b) $H_{GA}$, N = 500, f = 1




(c) $H_B$, N = 200, f = 2

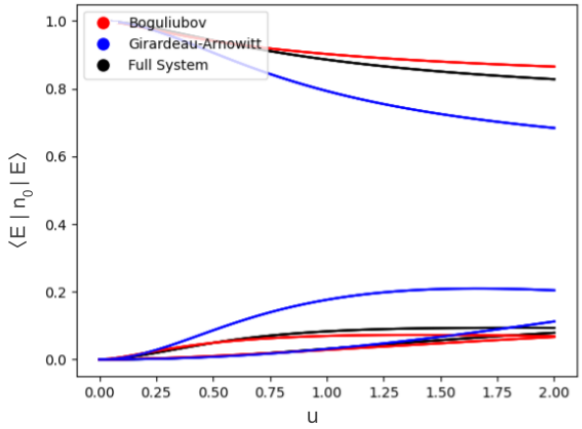



(d) $H_{GA}$, N = 200, f = 2

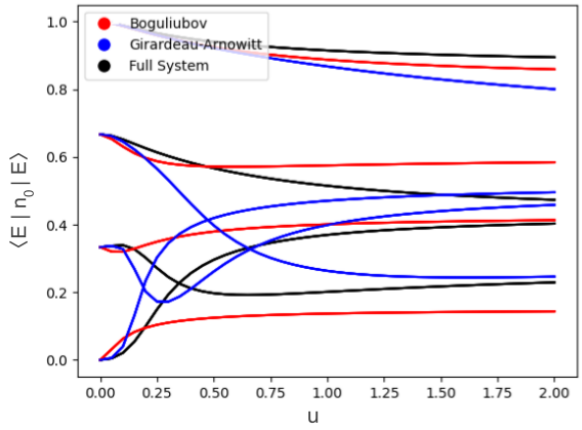
Figure 8: Histogram of consecutive level spacings shows that the level-spacing distribution decays rapidly for $u > 1$ in the many mode systems. The GA system exhibits a strangely uniform LSD for $f = 1$, however.

The markedly different behavior of the $f = 1$ and $f > 1$ systems is also apparent in the level-spacing distributions (LSD), which is defined as the distribution of differences between consecutive eigenvalues of the Hamiltonian, when ordered from smallest to largest. In Figure 8, we calculate the LSDs of the

Boguliubov and GA Hamiltonians at several values of $u$. For larger couplings, the energy gaps decay more rapidly. In the non-interacting limit $u = 0$, spacings are regular, although for many modes there is a higher degree of degeneracy, which explains the spike around 0. The most unexpected feature is the relatively uniform 'picket-fence' distribution exhibited by the energy levels of $H_{GA}$ when $f = 1$. However, we stress that the distributions plotted above are particularly sensitive to the dimensionality of the system. Even in 2 dimensions, the $u = 0$ limit would no longer appear as a single spike. (See, for instance, [6].)




(a) Condensate Fraction for N = 2, f = 2




(b) Condensate fraction for N = 6, f = 1

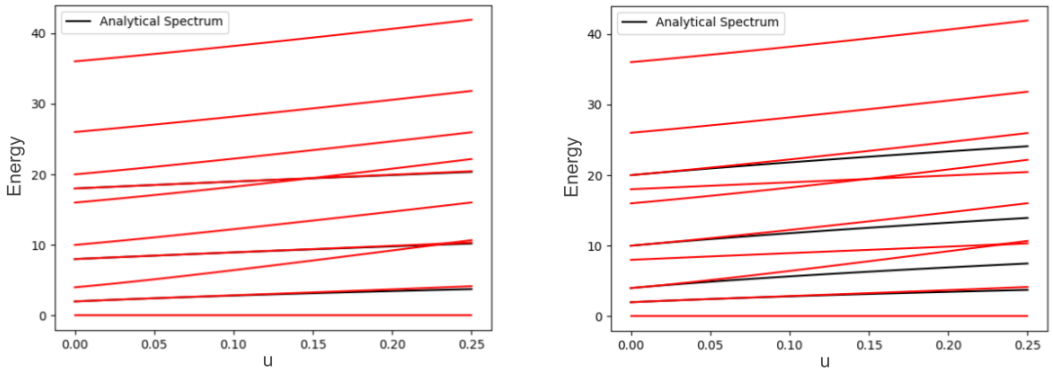
Figure 9: Eigenstate condensate fraction vs. $u$ for the three models. Upper curves correspond to the ground state, in which the condensate is most heavily occupied.

Apart from the spectrum, we may study the structure of the underlying eigenstates by various expectation values. For each eigenstate $|E\rangle$, we calculate

the 'condensate fraction':

$$\langle E|\frac{\hat{n}_0}{\hat{N}}|E\rangle$$

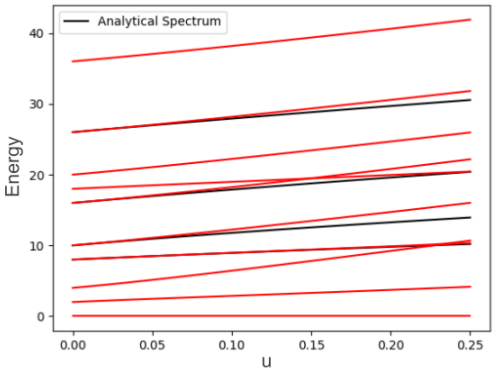
Figure 9 depicts similar depletion of the condensate in the ground state of each system as the coupling strength increases.



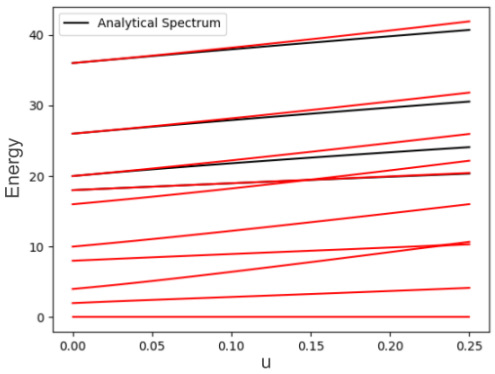

(a) $E(k)$ (no offset)




(b) $E(1) + E(k)$




(c) $E(2) + E(k)$




(d) $E(3) + E(k)$

Figure 10: Energies of elementary excitation pairs $E(k) = 2\sqrt{k^4 + 2uNk^2}$ plotted against the numerical spectrum (in red), each with a different elementary 'offset.'

The persistent questions underlying these numerical investigations continue to be the existence of conserved quantities which diagonalize the Hamiltonian. For instance, to construct a dispersion relation for $H_{GA}$, as Boguliubov did for $H_B$, would require an appropriate choice of basis relabelled by $K$. Absent such a transformation, it is difficult to decompose spectra into a set of 'elementary

excitations.' Nonetheless, we can see agreement between the analytical spectrum derived by Boguliubov, and the numerical spectrum obtained by exact diagonalization, as seen in Figure 10. Thus, exact diagonalization not only remains an interesting playground to explore properties of the quantum system, but provides a consistency check for any analytical results about the spectrum.

# 5 Rethinking the Quasilinear Analogy

Thus far, our discussion has only hinted at an analogy between the quasilinear approximation and the quantum gas. However, the connection can be made more explicit if we understand the mean-field theory of the Bose gas as a classical field theory with an associated hydrodynamic equation. In this regime, we replace the local field operator with a complex scalar field $\hat{\psi}(\vec{x},t) \mapsto \psi(\vec{x},t)$ [25]. Then the Hamiltonian becomes:

$$H[\psi^*,\psi] = \int_\Omega \frac{\hbar^2}{2m}|\nabla\psi|^2 + \frac{u}{2}|\psi|^4 - \mu|\psi|^2\,dx \tag{5.1}$$

In the above, $\mu$ is introduced as a Lagrange multiplier which fixes the particle number $N$. Setting $\hbar = 1$, we find the ground state of the gas satisfies Hamilton's equation:

$$i\dot{\psi} = \frac{\delta H}{\delta\psi} \tag{5.2}$$

Which, in this case, yields the Gross-Pitaevskii equation (GPE), known to mathematicians as the nonlinear Schrodinger equation:

$$i\dot{\psi} = \{\frac{1}{2m}\Delta + u|\psi|^2\}\psi \tag{5.3}$$

See [21] for a quick derivation of GPE from the many-body Schrodinger equation. Consider a stationary solution to the GPE, which satisfies the nonlinear eigenvalue equation:

$$\mu\psi_0 = \{\frac{1}{2m}\Delta + u|\psi_0|^2\}\psi_0$$

In the case of a homogenous BEC, we obtain a constant ground-state solution to the above equation, given a large enough box (and no trapping potential). Expanding to lowest orders of a perturbation about such a solution, we can actually derive a set of plane-wave solutions which obey the Boguliubov dispersion relation [25].

In what follows, we sketch a suggestive calculation which demonstrates how the coherent-state equations for the Boguliubov and Girardeau-Arnowitt systems can be seen as a quasilinear-type approximation to the GPE, although they each satisfy Hamilton's equations for some functional.

First let us write the Fourier expansion:

$$\psi(\vec{x},t) = \sum_k \phi_k e^{ik\cdot x} = \psi_0(t) + \psi_1(\vec{x},t)$$

Where $\psi_0 = \phi_0$ is a spatial average over the field $\psi$ and $\psi_1$ is a sum over non-condensate terms. If we assume $\frac{d\psi_0}{dt} = 0$, then we can apply a global U(1) transformation to assume $\psi_0 = \sqrt{n_0}$ is a real constant.

$$H_B[\psi_1^*, \psi] = \int \frac{1}{2m}|\nabla\psi_1|^2 + un_0|\psi_1|^2 + \frac{un_0}{2}(\psi_1^2 + (\psi_1^*)^2)\, dx \tag{5.4}$$

However, if we consider $\psi_0$ to be a complex function of time we obtain the Girardeau-Arnowitt Hamiltonian:

$$H_{GA}[\psi_1^*, \psi_0^*, \psi_1, \psi_0] = \int \frac{1}{2m}|\nabla\psi_1|^2 + u|\psi_0|^2|\psi_1|^2 + \frac{u}{2}((\psi_0^*)^2\psi_1^2 + \psi_0^2(\psi_1^*)^2)\, dx \tag{5.5}$$

These functionals are the quadratic terms in $\psi_1^{(*)}$ when substituting $\psi = \psi_0 + \psi_1$ into the fully-interacting Hamiltonian. Substituting the Fourier expansion, we obtain the familiar coherent state expressions (up to constant terms) for $H_{GA}$ and $H_B$.

Let us now consider a direct Fourier expansion of the time-dependent GPE 5.3. Matching Fourier coefficients, we find:

$$i\dot{\phi}_0 = \sum_{k_1,k_2,k_3} \phi_{k_1}\phi_{k_2}\phi_{k_3}^*\delta_{k_1+k_2,k_3} \tag{5.6}$$

$$= u|\phi_0|^2\phi_0 + 2u\phi_0\sum_{k\neq 0}\phi_k\phi_k^* + u\phi_0^*\sum_{k\neq 0}\phi_k\phi_{-k} + \sum_{k_1,k_2,k_3\neq 0}\phi_{k_1}\phi_{k_2}\phi_{k_3}^*\delta_{k_1+k_2,k_3} \tag{5.7}$$

$$= u(N+N_{nc})\phi_0 + u\phi_0^*\sum_{k\neq 0}\phi_k\phi_{-k} + \sum_{k_1,k_2,k_3\neq 0}(\dots) \tag{5.8}$$

For the non-condensate modes, we find

$$i\dot{\phi}_k = \epsilon_k\phi_k + \sum_{k_1,k_2,k_3}\phi_{k_1}\phi_{k_2}\phi_{k_3}^*\delta_{k_1+k_2,k_3+k} \tag{5.9}$$

$$= \epsilon_k\phi_k + u\phi_k\big(-|\phi_k|^2 + \sum_m|\phi_m|^2\big) + u\phi_0^2\phi_{-k}^* + \sum_{k_1,k_2\neq\{0,k\}}\phi_{k_1}\phi_{k_2}\phi_{k_3}^*\delta_{k_1+k_2-k_3,k} \tag{5.10}$$

$$= (\epsilon_k + 2uN - u|\phi_k|^2)\phi_k + u\phi_0^2\phi_k^* + \sum_{k_1,k_2\neq\{0,k\}}(\dots) \tag{5.11}$$

Note the striking resemblance between these equations and 4.2. Just as the Boguliubov theory can be obtained from an expansion of $H[\psi^*,\psi]$ in terms of $\psi = \psi_0 + \psi_1$, we can similarly see the coherent state equations of motion

following from a Fourier expansion of the GPE. Similar equations appear in the study of turbulence and kinetics in BEC and quantum gases. See for instance, [25]. However, more work must be done to sort out the discrepancies between the various formulas discussed above and to determine the validity of the pair approximation scheme in understanding BEC dynamics.

# 6 Conclusion

In this work, we examined the basic properties of the particle number-conserving theory of the dilute Bose gas proposed by Girardeau and Arnowitt. Using Noether's theorem, we demonstrated that the conserved quantities $\hat{n}_k - \hat{n}_{-k}$ are associated with the reduced U(1) symmetry in both the Boguliubov and GA theories. This argument clarified that the reduced U(1) symmetry does *not* imply the exact solvability of the Boguliubov model. Rather, the Boguliubov transform enables the realization of an infinite U(1) symmetry which does yield an exactly solvable model.

A further set of questions then relate to the idea of integrability. For the classical dynamical system obtained from the Lagrangian, complete integrability remains a promising path to understand the structure of the underlying quantum model. On the quantum end, there is a substantial body of literature on the subject of 'quantum integrability.'

In some cases, this term is used to describe exactly solvable models, or those, such as the Lieb-Liniger model, which may be solved by a Bethe ansatz [15]. However, there are other definitions which seek to establish the analogy with classical dynamics more explicitly, such as the existence of a maximal set of independent commuting quantities. Yet, there are several subtleties regarding the notion of 'maximal' and 'independent' in the quantum case.

For any quantum mechanical system, there are always a trivial set of conserved quantities which diagonalize the Hamiltonian. If $\{|E_i\rangle\}$ are the eigenstates of $H$, then the projection operators $|E_i\rangle \langle E_i|$ are conserved quantities.

There are further connections to be made with the existence of level-crossings, and the statistics of level-spacings (i.e. the difference between consecutive eigenvalues). For instance, the Berry-Tabor conjecture predicts that the level-spacing distribution is Poisson, for systems which are integrable in their classical limit. See [5] for a comparison these different notions of integrability.

The deeper question then remains, is the GA system 'integrable' and in what sense? We pursued two distinct approaches to investigating hidden symmetries/conservation laws by restricting to a finite number of excited modes. With only 1 excited mode, we offered strong evidence for complete integrability of the corresponding classical system. An exact analytical solution may be possible by some change of variables, and would offer clues as to whether the $f = 1$ case is special or a many-mode generalization is possible.

Beyond the $f = 1$ case, a more serious numerical investigation into the system with several excited modes, particularly the presence of Hamiltonian chaos, would offer a strong indicator for the lack of integrability. The implications for

the underlying quantum system will require further consideration.

Moreover, it could be interesting to study the semi-classical limit of the functional integral developed in 2.3, by applying the stationary phase method and expanding in the vicinity of the solution obtained by the Euler-Lagrange equations.

On the matrix representation side, there are several further steps that could be taken to broaden the scope of the investigation. First, the code used may be generalized to include systems of 2 or 3 dimensions, although when computing matrix diagonalizations, a more efficient scheme may be desired due to the large complexity of the problem.

Theoretical insights will be necessary to go about constructing a numerical dispersion relation for the GA system. While we have developed the numerical apparatus to examine level-crossings and level-spacing distributions, the task of interpreting these indicators is not easy. For instance, the fully-interacting (Lieb-Liniger) model in 1-dimension is itself integrable, but the presence of Poisson statistics in the LSD depends on the choice of subspace in which the eigenvalues are computed [16].

While this thesis leaves far more to be explored about the specific models it addresses, the connections to the classical quasilinear approximation also suggests exciting parallels in the study of quantum and classical fluids, each of which manifest turbulence and strange, collective behaviors.

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# 8 Appendix: Code

## 8.1 Coherent State ODEs

The following code was written in the Julia programming language:

```
    using Plots
using DifferentialEquations



# Girardeau-Arnowitt EOMs
# As presented in the paper
function modeNsys!(d::Array{ComplexF64},::Array{ComplexF64}, param, t)
    n, N, u = param
    N_c = abs2([1])
    N_nc = N - N_c
    d[1] = -im*u*(N -.5 + N_nc)*[1]
    for i in 1:n
        d[1] += -im*u*2*[2*i]*[2*i + 1]*conj([1])
        d[2*i] = -im*u*((N -.5 + N_c + i^2/u)*[2*i] + conj([2*i + 1])*([1])^2)
        d[2*i + 1] = -im*u*((N -.5 + N_c + i^2/u)*[2*i + 1] + conj([2*i])*([1])^2)
    end
end

# Modified EOMs when uN(N-.5) is assumed to be a pure constant
function modeNsysv2!(d::Array{ComplexF64},::Array{ComplexF64}, param, t)
    n, N, u = param
    N_c = abs2([1])
    N_nc = N - N_c
    d[1] = -im*u*(N_nc + 10)*[1]
    for i in 1:n
        d[1] += -im*u*2*[2*i]*[2*i + 1]*conj([1])
        d[2*i] = -im*u*((N_c + 10 + i^2/u)*[2*i] + conj([2*i + 1])*([1])^2)
        d[2*i + 1] = -im*u*((N_c + 10 + i^2/u)*[2*i + 1] + conj([2*i])*([1])^2)
    end
end

# Boguliubov EOMs w/ n excited modes
function bogNsys!(d::Array{ComplexF64},::Array{ComplexF64}, param, t)
    n, N, u = param
    for i in 1:n
        d[2*i - 1] = im*(-(i^2 + u*N)*[2*i - 1] + u*N*conj([2*i]))
        d[2*i] = im*(-(i^2 + u*N)*[2*i] + u*N*conj([2*i - 1]))
    end
end
```

```
# Returns total particle number
function partNum()
    N = 0.0
    for i in eachindex()
        N += abs2([i])
    end
    return N
end

# Returns n conserved quantities associated with difference in Momentum states
function mode_diff()
    n = (length() - 1) ÷ 2
    empty_arr = []
    for i in n
        push!(empty_arr, abs2([2*i]) - abs2(2*i + 1))
    end
    return empty_arr
end

# Returns energy of a given state
function energy(, params)
    n, N, u = params
    E = 0
    for i in 1:n
        E += (i^2 + u*abs2([1]))*(abs2([2*i]) + abs2([2*i+1]))
        E += 2*u*real([2*i]*[2*i+1]*conj([1])^2)
    end
    return E
end

# Implements Boguliubov EOMs
function runBog()
    n = 3
    N = 6
    u = 1.0
     = [1.0 + 0*im, 1.0+ 0.0*im, 1.0 + 1*im, 0.0 + 0*im, 0 + 1*im, 1 + 0*im]
    # initial condition in phase space
    t_final = 50.0 # final time of simulation
    params = (n, N, u)
    tspan = (0.0, t_final) # span of time to simulate

    prob = ODEProblem(bogNsys!, , tspan, params) # specify ODE
    sol = solve(prob, Tsit5(), reltol=1e-12, abstol=1e-12)
    print("Initial Differences:  ")
    print([abs2([1])- abs2([2]), abs2([3])- abs2([4]), abs2([5])- abs2([6])])
    print("   Final Differences:  ")
```

```
    print([abs2(sol[1, end]) - abs2(sol[2, end]),
    abs2(sol[3, end])- abs2(sol[4, end]), abs2(sol[5, end])- abs2(sol[6, end])])
    sample_times = sol.t
    #plot(sample_times, abs2.(sol[1,:]), color = "black", legend = false)
    #plot!(sample_times, abs2.(sol[3,:]), color = "green")
    #plot!(sample_times, abs2.(sol[5,:]), color = "red")
    plot(real.(sol[1,:]), imag.(sol[1,:]), aspect_ratio=:equal, label = "")
    plot!(real.(sol[3,:]), imag.(sol[3,:]), aspect_ratio=:equal,label = "")
    plot!(real.(sol[5,:]), imag.(sol[5,:]), aspect_ratio=:equal,label = "")
    xlabel!("Re[]")
    ylabel!("Im[]")
    savefig("Bog $params .png")
end

# INITIAL CONDITIONS
n = 3
# = [5 + 0*im, 1.0 + 0*im, 1.0+ 0.0*im, 1.0 + 0*im, 0.0 + 0*im, 0 + 0.5*im, .1 + 0*im]
 = [5.0 + 0*im, 0 + 1.0*im, 1.0+ 0.0*im, 1 + 0*im, 1.0 + 0*im, 0 + .5*im, .1 + 1*im]
# initial condition in phase space
# = [5 + 0*im, 1.0 + 0*im, 0.0+ 0.0*im, 0.25 + 0*im, 0.0 + 0*im, 0 + 0*im, 0 + 0*im]

t_final = 10.0 # final time of simulation
u = 0.25



# SOLVE GA THE PROBLEM


params = (n, partNum(), u)
print(params)


tspan = (0.0, t_final) # span of time to simulate

prob = ODEProblem(modeNsysv2!, , tspan, params) # specify ODE
sol = solve(prob, Tsit5(), reltol=1e-12, abstol=1e-12)

sample_times = sol.t


print("Initial particle num = $(partNum(sol[:,1]))   ")
print("Final particle num = $(partNum(sol[:,end]))   ")
print(sol[:,1])
print("     Initial energy    ")
print(energy(sol[:,1], params))
```

```
print("     Final Energy ")
print(energy(sol[:,end],params))

# Plots |phi|^2 for each mode

plot(sample_times, abs2.(sol[1,:]), color = "black", label = "||^2")
plot!(sample_times, abs2.(sol[2,:]), color = "green", label = "||^2")
plot!(sample_times, abs2.(sol[4,:]), color = "red", label = "||^2")
plot!(sample_times, abs2.(sol[6,:]), color = "yellow", label = "||^2")
xlabel!("t")
#savefig("4 modes $params .png")

# Plots phi_0 in the complex plane

#plot(real.(sol[1,:]), imag.(sol[1,:]), aspect_ratio=:equal, legend=false)
#xlabel!("Re[]")
#ylabel!("Im[]")
#savefig("50s 3mode argand $params .png")
```

## 8.2 Exact Diagonalization

The following code was written in Python:

```
import numpy as np
import sympy.physics.secondquant as sq
from sympy import *
init_printing(use_unicode=True)
from sympy.physics.secondquant import Dagger, B, BKet
from numpy import sqrt
import matplotlib.pyplot as plt
from itertools import combinations_with_replacement as comb

"N particles, f non-condensate degs of freedom"

# Computes matrix reps. of Boguliubov and Gardiner Hamiltonians

import numpy as np
import matplotlib.pyplot as plt
from numpy import sqrt
from sympy import *
init_printing(use_unicode=True)
from itertools import combinations_with_replacement as comb

N = 2 # number of particles
f = 1 # number of excited modes
u = symbols('u') # introduces coupling constant as a variable
```

```
# create a basis for N-particle subspace
def create_basis(N, f):
  l = range(f+1)
  comb_list = list(comb(l, N//2))
  basis = []
  for b in comb_list:
    new_state = []
    for i in l:
      new_state.append(b.count(i))
    new_state[0] = 2*new_state[0]
    basis.append(tuple(new_state))
  return basis

"""
Each basis state is denoted (n0, n1+, n2+, ...) where
n1+ is half the number of particles in the first excited mode.
Thus, N = n0 + 2*n1+ + 2*n2+ + ...
Also, n0 is always even as we only consider
states accessible to the state (N, 0, 0, ...)
"""

# computes <i|H|j> for basis elements i,j in B, G, and GA models

def matrix_elem(i,j, B, N, f):
  n0 = B[j][0]
  diff = n0 - B[i][0]
  tuple_diff = tuple(np.subtract(B[j],B[i])[1:])
  if j == i:
    # diagonal elements
    E0 = (u/2)*N*(N - 1)
    Eb,Eg, Ega = 0, 0,0
    for k in range(f):
      # sums over excited states
      Eg += ((k+1)**2/N + u)*B[j][k+1]
      Eb += ((k+1)**2 + u*N)*B[j][k+1]
      Ega += ((k+1)**2 + u*n0)*B[j][k+1]
    g = E0 + 2*(n0+1)*Eg
    ga = E0 + 2*Ega
    b = E0 + 2*Eb
  elif abs(diff) > 2 or tuple_diff.count(0) != (f-1):
    # under the quasilinear approximation these share no probability amplitude
    b, g, ga = 0, 0, 0
  elif diff == 2:
      m = (B[j] - B[i]).index(-1)
      # this is the mode into which two particles have scattered
```

```
      g = u*(B[j][m] + 1)*sqrt(n0*(n0 - 1))
      ga = g
      b = u*N*(B[j][m] + 1)
  else:
      tuple_diff = tuple(np.subtract(B[j],B[i]))
      m = tuple_diff.index(1)
      # two particles have scattered into the condensate from this mode
      g = u*(B[j][m])*sqrt((n0 + 1)*(n0 + 2))
      ga = g
      b = u*N*(B[j][m])
  return b, g, ga

#-------------------------------------------------------


# generates sympy matrix of Boguliubov and G-A hamiltonians
def compute_matrix(basis, N, f):
  s = len(basis)
  # the Hamiltonian will be an s x s matrix in this subspace
  b = zeros(s,s)
  ga = zeros(s,s)
  for i in range(s):
    # i = bra / row
    for j in range(i,s):
    # j = ket / col
      bij, gij, gaij = matrix_elem(i,j, basis, N,f)
      # matrix is symmetric so we fill upper/lower simultaneously
      b[i,j] = bij
      b[j,i] = bij
      ga[i,j] = gaij
      ga[j,i] = gaij
  return b, ga

# creates array from analytical spectrum
def exc(N,k):
  return sqrt(u*N*(k**2) + k**4)

def analytical_spec(N,f):
  E0 = (u/2)*N*(N-1)
  for i in range(f):
    E0 += (exc(N, i + 1) - u*N - (i + 1)**2) #- (u*N / (i+1))**2
  E_vals = [E0]
  for i in range(f):
    E_vals.append(E0 + exc(N, i+1))
  return E_vals
```

```
u = symbols('u')

# create basis for fully interacting system

def interacting_basis(N,f):
  l = range(-f, f+1)
  comb_list = list(comb(l, N))
  basis = []
  for b in comb_list:
    # the total momentum of system remains zero
    if sum(b) == 0:
      new_state = []
      new_state.append(b.count(0))
      for i in range(1, f+1):
        new_state.append(b.count(i))
        new_state.append(b.count(-1*i))
      basis.append(sq.FockStateBosonKet(new_state))
  return basis

"""
Basis states for interacting system are now denoted
by tuples of the form (n0, n1+, n1-, n2+, n2-,...)
n0 is not necessarily even, but we do require that:
(n1+ - n1-) + (n2+ - n2-) + ... = 0
"""

# input: k-value, output: corr. index in the basis vector
def index(i):
  if i <= 0:
    index = -2*i
  else:
    index = 2*i -1
  return index

# constructs full Hamiltonian
def fullHamiltonian(f):
  k_vals = range(-f, f+1)
  op = 0
  for i in k_vals:
    i_ = index(i)
    # one-body terms
    op += (i**2)*Dagger(B(i_))*B(i_)
    for j in k_vals:
      j_ = index(j)
      for k in k_vals:
        k_ = index(k)
```

```
        if (i + j - k) in k_vals:
          # two-body terms
          op += (u/2)*Dagger(B(i_))*Dagger(B(j_))*B(k_)*B(index(i + j - k))
  return op

# computes matrix rep. of H in given basis
def compute_full_matrix(N,f):
  basis = interacting_basis(N,f)
  op = fullHamiltonian(f)
  matrix = sq.matrix_rep(op, basis)
  return basis, matrix

"""NOTE: my reliance on sympy to compute these matrices
results in extremely slow computation. By directly
computing the matrix elements, a much faster way
to do this could maybe be achieved. """

# Below I plot spectra for Bog. and G-A and Full systems.

# I stored several matrices and bases for the full system, generated with the code above, ir

def generateSpectralComparison(N,f):

  basis = create_basis(N, f)
  b, ga = compute_matrix(basis, N, f)

  b_func = lambdify(u, b, modules='numpy')
  ga_func = lambdify(u, ga, modules='numpy')

  # Matrices of the fully-interacting Hamiltonian have...
  # separately been computed and stored in the dictionary full_dict
  f_key = "M" + str(N) + str(f)
  F_matrix = full_dict[f_key][0]

  u_vals = np.linspace(0, 2, 20)   # consider x values 0, 1, .., 100
  eigenvalsB = np.array([np.sort(np.linalg.eigvals(b_func(i))) for i in u_vals])
  eigenvalsG = np.array([np.sort(np.linalg.eigvals(ga_func(i))) for i in u_vals])
  eigenvalsF = np.array([np.sort(np.linalg.eigvals(F_matrix(i))) for i in u_vals])

  fig = plt.figure(figsize=(9,3))
  gs = fig.add_gridspec(1,3)
  axs = gs.subplots(sharex=True, sharey=True)
  fig.suptitle('Spectrum E(u) for N=%d, f = %d' % (N,f))
  for i in range(len(full_dict[f_key][1])):
    axs[2].plot(u_vals, eigenvalsF[:,i], color = 'black')
  for i in range(len(basis)):
```

```
    axs[0].plot(u_vals, eigenvalsB[:,i], color = 'red')
    axs[1].plot(u_vals, eigenvalsG[:,i],color = 'blue')

  handles = [plt.Line2D([0], [0], marker='o', color='w',
  markerfacecolor='red', markersize=10),  plt.Line2D([0], [0],
  marker='o', color='w', markerfacecolor='blue', markersize=10),
  plt.Line2D([0], [0], marker='o', color='w', markerfacecolor='black', markersize=10)]

  labels = ['Boguliubov', 'Girardeau-Arnowitt', 'Full System']

  # Place the legend in the upper left corner and specify the handles and labels
  plt.legend(handles=handles, labels=labels, loc='upper left')

  plt.savefig("3x1_SpecN" + str(N) + "f" + str(f) + ".png")
  plt.show()


# Computes expectation value of n_0/N w.r.t the given basis
def condensate_frac(vec, basis, N):
  sum = 0
  for i in range(len(basis)):
    sum += (vec[i]**2)*basis[i][0]
  return sum / float(N)

# Plots condensate fraction comparison over a range of u values
def condFracComparison(N,f):
  basis = create_basis(N,f)
  b, g = compute_matrix(basis, N, f)
  b_func = lambdify(u, b, modules='numpy')
  g_func = lambdify(u, g, modules='numpy')
  u_vals = np.linspace(0, 2, 41)   # consider x values 0, 1, .., 100
  f_key = "M" + str(N) + str(f)
  F_matrix = full_dict[f_key][0]
  F_basis = full_dict[f_key][1]
  eigvalsB_u = []
  eigvalsG_u = []
  eigvalsF_u = []
  condFracB_u = []
  condFracG_u = []
  condFracF_u = []
  for u_val in u_vals:
    eigvalsB, eigvecsB = np.linalg.eigh(b_func(u_val))
    eigvalsG, eigvecsG = np.linalg.eigh(g_func(u_val))
    eigvalsF, eigvecsF = np.linalg.eigh(F_matrix(u_val))
    condFracB = []
    condFracG = []
```

```
    condFracF = []
    orderB = np.argsort(eigvalsB)
    orderG = np.argsort(eigvalsG)
    orderF = np.argsort(eigvalsF)
    for i in range(len(basis)):
      condFracB.append(condensate_frac(eigvecsB[:,i],basis, N))
      condFracG.append(condensate_frac(eigvecsG[:,i],basis, N))
    for i in range(len(eigvecsF)):
      condFracF.append(condensate_frac(eigvecsF[:,i],F_basis, N))
    condFracB = np.array(condFracB)[orderB]
    condFracG = np.array(condFracG)[orderG]
    condFracF = np.array(condFracF)[orderF]
    condFracB_u.append(condFracB)
    condFracG_u.append(condFracG)
    condFracF_u.append(condFracF)
  condFracB_u = np.array(condFracB_u)
  condFracG_u = np.array(condFracG_u)
  condFracF_u = np.array(condFracF_u)
  fig = plt.figure(figsize=(9,3))
  gs = fig.add_gridspec(1,3, width_ratios=[2, 2, 2])
  axs = gs.subplots(sharex=True, sharey=True)
  fig.suptitle('Condensate Fraction for N=%d, f = %d' % (N,f))
  for i in range(len(full_dict[f_key][1])):
    axs[2].plot(u_vals, condFracF_u, color = 'black')
  for i in range(len(basis)):
    axs[0].plot(u_vals, condFracB_u, color = 'red')
    axs[1].plot(u_vals, condFracG_u,color = 'blue')

  handles = [plt.Line2D([0], [0], marker='o', color='w', markerfacecolor='red',
  markersize=10), plt.Line2D([0], [0], marker='o', color='w',
  markerfacecolor='blue', markersize=10), plt.Line2D([0], [0], marker='o',
  color='w', markerfacecolor='black', markersize=10)]
  labels = ['Boguliubov', 'Girardeau-Arnowitt', 'Full System']

  # Place the legend in the upper left corner and specify the handles and labels
  plt.legend(handles=handles, labels=labels, loc='upper left')

  plt.savefig("3x1_CF_N" + str(N) + "f" + str(f) + ".png")
  plt.show()
```

The remaining section is a piece of more efficient code that computes the Boguliubov and Girardeau-Arnowitt systems without Sympy.

```
    "N particles, f non-condensate degs of freedom"

# Computes matrix reps. of Boguliubov and Girardeau-Arnowitt Hamiltonians
```

```
# create a basis for N-particle subspace
def create_basis(N, f):
  l = range(f+1)
  comb_list = list(comb(l, N//2))
  basis = []
  for b in comb_list:
    new_state = []
    for i in l:
      new_state.append(b.count(i))
    new_state[0] = 2*new_state[0]
    basis.append(tuple(new_state))
  return basis

"""
Each basis state is denoted (n0, n1+, n2+, ...) where n1+ is half the number
of particles in the first excited mode.
Thus, N = n0 + 2*n1+ + 2*n2+ + ...
Also, n0 is always even as we only consider states accessible to the state (N, 0, 0, ...)
"""

# computes <i|H|j> for basis elements i,j in B and GA models

def matrix_elem_v2(i,j, B, N, f):
  n0 = B[j][0]
  diff = n0 - B[i][0]
  tuple_diff = tuple(np.subtract(B[j],B[i])[1:])
  ga_0 = 0
  b_0 = 0
  if j == i:
    # diagonal elements
    E0 = 0.5*N*(N - 1) # *u
    Eb,Ega = 0, 0
    # Eg = 0
    E_k = 0
    for k in range(f):
      # sums over excited states
      E_k += ((k+1)**2)*B[j][k+1]
      Eb += N*B[j][k+1]
      Ega += n0*B[j][k+1]
    ga_0 = 2*E_k
    ga_1 = E0 + 2*Ega
    b_0 = 2*E_k
    b_1 = E0 + 2*Eb
  elif abs(diff) > 2 or tuple_diff.count(0) != (f-1):
```

```
    # under the quasilinear approximation these share no probability amplitude
    b_1, ga_1 = 0, 0
  elif diff == 2:
      m = (B[j] - B[i]).index(-1)
      # this is the mode into which two particles have scattered
      ga_1 = (B[j][m] + 1)*sqrt(n0*(n0 - 1))
      b_1 = N*(B[j][m] + 1)
  else:
      tuple_diff = tuple(np.subtract(B[j],B[i]))
      m = tuple_diff.index(1)
      # two particles have scattered into the condensate from this mode
      ga_1 = (B[j][m])*sqrt((n0 + 1)*(n0 + 2))
      b_1 = N*(B[j][m])
  return b_0, b_1, ga_0, ga_1

#-------------------------------------------------------


# generates sympy matrix of Boguliubov and G-A hamiltonians
def compute_matrix(basis, N, f):
  s = len(basis)
  # the Hamiltonian will be an s x s matrix in this subspace
  b = zeros(s,s)
  ga = zeros(s,s)
  for i in range(s):
    # i = bra / row
    for j in range(i,s):
    # j = ket / col
      bij, gij, gaij = matrix_elem(i,j, basis, N,f)
      # matrix is symmetric so we fill upper/lower simultaneously
      b[i,j] = bij
      b[j,i] = bij
      ga[i,j] = gaij
      ga[j,i] = gaij
  return b, ga

def compute_matrix_v2(basis, N, f):
  s = len(basis)
  # the Hamiltonian will be an s x s matrix in this subspace
  b0 = np.zeros((s,s))
  ga0 = np.zeros((s,s))
  b1 = np.zeros((s,s))
  ga1 = np.zeros((s,s))
  for i in range(s):
    # i = bra / row
    for j in range(i,s):
```

```
    # j = ket / col
      b0ij,b1ij, ga0ij, ga1ij = matrix_elem_v2(i,j, basis, N,f)
      # matrix is symmetric so we fill upper/lower simultaneously
      b0[i,j] = b0ij
      b0[j,i] = b0ij
      b1[i,j] = b1ij
      b1[j,i] = b1ij
      ga0[i,j] = ga0ij
      ga0[j,i] = ga0ij
      ga1[i,j] = ga1ij
      ga1[j,i] = ga1ij
  return b0, b1, ga0, ga1

# creates array from analytical spectrum
def exc(u,N,k):
  return sqrt(2*u*N*(k**2) + k**4)

def ground_state(u,N,f):
  E0 = (u/2)*N*(N-1)
  for i in range(f):
    E0 += 0.5*(exc(u, N, i + 1) - u*N - (i + 1)**2 + (u*N / (i+1))**2)
  return E0

# spectralDist plots level-spacing distributions
def spectralDist_v2(N,f):
  basis = create_basis(N,f)
  num = len(basis)
  numbins = 3*int(round(1 + 3.3*np.log10(num)))
  b0, b1, ga0, ga1 = compute_matrix_v2(basis, N, f)
  uvals = [0.0, 0.5, 1.0, 5.0]
  fig1, ax1 = plt.subplots(2,2)
  #figa, axa = plt.subplots(2,2)
  for i in range(4):
    eigenvalsB0 = np.sort(np.linalg.eigvalsh(ga0+uvals[i]*ga1))
    level_diff = np.subtract(eigenvalsB0[1:], eigenvalsB0[0:-1])
    j = 0
    k = i % 2
    if i >= 2:
      j = 1
    ax1[j,k].hist(level_diff, numbins)
    ax1[j,k].set_title("u = " + str(uvals[i]))
    #axa[j,k].hist(level_ratio)
  fig1.tight_layout()
  #figa.tight_layout
  plt.savefig("GA_LSD_N200f2.png")
```

```
fig2, ax2 = plt.subplots(2,2)
for i in range(4):
  eigenvalsB0 = np.sort(np.linalg.eigvalsh(b0+uvals[i]*b1))
  level_diff = np.subtract(eigenvalsB0[1:], eigenvalsB0[0:-1])
  j = 0
  k = i % 2
  if i >= 2:
    j = 1
  ax2[j,k].hist(level_diff,numbins)
  ax2[j,k].set_title("u = " + str(uvals[i]))
fig2.tight_layout()
plt.savefig("B_LSD_N200f2.png")
```